

SYMMETRIC DESIGNS AND FINITE SIMPLE EXCEPTIONAL GROUPS OF LIE TYPE

SEYED HASSAN ALAVI, MOHSEN BAYAT, AND ASHARF DANESHKHAH

ABSTRACT. In this article, we study symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group G whose socle is a finite simple exceptional group of Lie type. We prove a reduction theorem to some possible parameters of such designs. In particular, we show that there is no such design for $\lambda \leq 87$. As a main tool to this investigation, part of this paper is devoted to studying large maximal subgroups of almost simple groups whose socle is a finite simple exceptional group of Lie type.

1. INTRODUCTION

A t -(v, k, λ) design $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is an incidence structure consisting of a set \mathcal{V} of v points, and a set \mathcal{B} of k -element subsets of \mathcal{V} , called *blocks*, such that every t -element subset of points lies in exactly λ blocks. The design is *nontrivial* if $t < k < v - t$, and is *symmetric* if $|\mathcal{B}| = v$. By [6, Theorem 1.1], if \mathcal{D} is symmetric and nontrivial, then $t \leq 2$ (see also [15, Theorem 1.27]). Thus we study nontrivial symmetric 2-(v, k, λ) designs which we simply call *symmetric (v, k, λ) designs*. A *flag* of \mathcal{D} is an incident pair (α, B) where α and B are a point and a block of \mathcal{D} , respectively. An *automorphism* of a symmetric design \mathcal{D} is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of \mathcal{D} is called *flag-transitive* if it is transitive on the set of flags of \mathcal{D} . If G is primitive on the point set \mathcal{V} , then G is said to be *point-primitive*.

It is known that if a nontrivial symmetric (v, k, λ) design \mathcal{D} with $\lambda \leq 100$ admitting a flag-transitive, point-primitive automorphism group G , then G must be an affine or almost simple group [44]. Therefore, it is interesting to study such designs whose socle is of almost simple type or affine type with large λ . In this direction, it is recently shown in [1] that there are only five possible symmetric (v, k, λ) designs (up to isomorphism) admitting a flag-transitive and point-primitive automorphism group G satisfying $X \trianglelefteq G \leq \text{Aut}(X)$ where $X = PSL_2(q)$, see also [46]. In the case where X is a sporadic simple group, there also exist four possible parameters (see [45]). This study for $X := PSL_3(q)$ gives rise to one nontrivial design which is a Desarguesian projective plane $PG_2(q)$ and $PSL_3(q) \leq G$ (see [2]), however when $X = PSU_3(q)$, there is no such non-trivial symmetric designs (see [10]). This paper is devoted to studying symmetric designs admitting a flag-transitive and point-primitive almost simple automorphism group G whose socle is a finite simple exceptional group of Lie type.

Date: February 7, 2017.

1991 Mathematics Subject Classification. 05B05; 05B25; 20B25.

Key words and phrases. Flag-transitive; point-primitive; automorphism group; symmetric design; finite simple exceptional group; large subgroups.

Corresponding author: S.H. Alavi.

Theorem 1.1. Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a nontrivial symmetric (v, k, λ) design with $\lambda \geq 4$, and let G be a flag-transitive and point-primitive automorphisms group of \mathcal{D} whose socle X is a finite simple exceptional group of Lie type. Let also $H = X_\alpha$ with $\alpha \in \mathcal{V}$. Then

- (a) if $X = G_2(q)$, then $X \cap H$ and (v, k, λ) are as in Table 1;
- (b) if $X \neq G_2(q)$, then H is non-parabolic and (X, H) is as in Table 2.

TABLE 1. The parameters in Theorem 1.1 if $X = G_2(q)$ with $q = p^a$.

Line	$X \cap H$	v	k	λ	Comments
1	$[q^5] : \text{GL}_2(q)$	$\frac{(q^6-1)}{q-1}$	q^5	$q^4(q-1)$	parabolic
2	$SL_3^\epsilon(q) \cdot 2$	$\frac{q^3(q^3+\epsilon 1)}{q-1}$	$\frac{(q^3-\epsilon 1)(q^3+\epsilon 2)+4}{4}$	$\frac{(q^3-\epsilon 1)(q^3+\epsilon 2)+4}{4}$	q odd
3	$SL_3^\epsilon(q) \cdot 2$	$\frac{q^3(q^3+\epsilon)}{2}$	$\frac{c(q^3-\epsilon 1)(q^3+\epsilon 2)+4d_\epsilon^2}{4d_\epsilon^2}$	$\frac{c^2(q^3-\epsilon 1)(q^3+\epsilon 2)+4cd_\epsilon^2}{8d_\epsilon^4}$	$q = 2^a \geq 8$, $c = 2, 2d_\epsilon$ with $d_+ = 3$ and $d_- = 5$
4	$SL_3^\epsilon(q) \cdot 2$	$\frac{q^3(q^3+\epsilon)}{2}$	$\frac{q^6-1}{q^3(q^3-\epsilon)}$	$\frac{2(q^3+\epsilon 1)(q^3-\epsilon 2)}{9}$	q odd and $3 \mid q + \epsilon 1$
5	$SL_3^\epsilon(q) \cdot 2$	$\frac{q^3(q^3+\epsilon)}{2}$	$\frac{q^3(q^3-\epsilon)}{q^3(q^3-\epsilon)}$	$\frac{q^3(q^3-\epsilon 3)}{9}$	$q = 3^a$
6	$SL_3^+(q) \cdot 2$	$\frac{q^3(q^3+1)}{2}$	$\frac{(q^3-1)(q^3-8)}{30}$	$\frac{(q^3-8)(q^3-11)}{450}$	$q = 2^a$ with a odd
7	$SL_3^-(q) \cdot 2$	$\frac{q^3(q^3-1)}{2}$	$\frac{(q^3+1)(q^3+4)}{18}$	$\frac{(q^3+4)(q^3+7)}{162}$	$q = 2^a$ with a odd
8	$SL_3^-(q) \cdot 2$	$\frac{q^3(q^3-1)}{2}$	$\frac{(q^3+1)^3}{9}$	$\frac{(q^3+1)(q^3+4)}{81}$	$q = 2^a$ with a odd
9	$SL_3^-(q) \cdot 2$	$\frac{q^3(q^3-1)}{2}$	$\frac{x^3(q^3+1)}{6}$	$\frac{q^3(q^3+3)}{18}$	$q = 2^a$ with a odd

TABLE 2. The parameters for Theorem 1.1 if $X \neq G_2(q)$.

Line	X	H	Conditions	Details on (p, a)
1	$F_4(q)$	$d^2 \cdot D_4(q) \cdot \text{Sym}_3$	$q \leq 2^{16}$	Table 13
2		${}^3D_4(q) \cdot 3$	$q \leq 2^{14}$	Table 14
3	$E_6^\epsilon(q)$	$d \cdot (A_1(q) \times A_5^\epsilon(q)) \cdot d \cdot e$	$(e, \epsilon) = (1, +)$, $q \leq 2^{15}$	Table 15
4			$(e, \epsilon) = (1, -)$, $q \leq 2^{16}$	Table 15
5			$(e, \epsilon) = (3, \pm)$, $q \leq 2^{21}$	Table 16
6		$h \cdot (D_5^\epsilon(q) \times (\frac{q-\epsilon 1}{h})) \cdot h$	$(e, \epsilon) = (1, \pm)$, $q \leq 2^{95}$	Table 17
7			$(e, \epsilon) = (3, \pm)$, $q \leq 2^{105}$	Table 17
8		$C_4(q) \cdot t$	q odd and $t \mid 2a$	
9			$(e, \epsilon) = (1, \pm)$, $q \leq 3^{22}$	Table 18
10			$(e, \epsilon) = (3, +)$, $q \leq 3^{30}$	Table 18
11			$(e, \epsilon) = (3, -)$, $q \leq 3^{29}$	Table 18
12		$F_4(q) \cdot t$	$t \mid \gcd(2, p)a$	
13	$E_7(q)$	$d \cdot (A_1(q) \times D_6(q)) \cdot d$	$q \leq 2^{19}$	Table 19
14		$h_1 \cdot (A_7^\epsilon(q) \cdot g \cdot (2 \times (2/h_1)))$	$q \leq 2^{17}$	Table 20
15		$e(E_6^\epsilon(q) \times (q - \epsilon/e)) \cdot e \cdot 2$	$q = 2^a$ with $a \leq 45$	Table 21
16	$E_8(q)$	$d \cdot (A_1(q) \times E_7(q)) \cdot d$	$q \leq 2^7$	Table 22

Note: $q = p^a$, $d = \gcd(2, q-1)$, $e = \gcd(3, q-\epsilon 1)$, $h = \gcd(4, q-\epsilon 1)$, $h_1 = \gcd(4, q-\epsilon 1)/\gcd(2, q-1)$, $g = \gcd(8, q-\epsilon 1)/\gcd(2, q-1)$ with $\epsilon = \pm$.

Remark 1.2 (Comments on Theorem 1.1).

- (a) The symmetric designs with $\lambda \leq 3$ and automorphisms groups satisfying the conditions in Theorem 1.1 have been studied in [39, 41, 50], and so in this paper we are interested in the case where $\lambda \geq 4$.
- (b) For small q , the parameters arose in Tables 1 and 2 are really large, however, if $\gcd(v-1, |H|)$ is small enough, then we can use Algorithm 1 based on Lemmas 2.4 and 2.6 to check if there is a possible symmetric design. Based on our observation, the existence of such possible designs depends on how much the parameter $\gcd(v-1, |H|)$ is large. We believe that the conditions in these tables give no symmetric designs satisfying the conditions in Theorem 1.1.

- (c) One way to study the remaining groups in Tables 1 and 2 is to investigate the suborbits of these groups which is part of our future research project.

Symmetric designs with λ small have been of most interest. Kantor [17] classified flag-transitive symmetric $(v, k, 1)$ designs (projective planes) of order n and showed that either \mathcal{D} is a Desarguesian projective plane and $PSL_3(n) \leq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, where n is even and $n^2 + n + 1$ is prime. Regueiro [35] gave a complete classification of biplanes ($\lambda = 2$) with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group (see also [36, 37, 38, 39]). Zhou and Dong studied nontrivial symmetric $(v, k, 3)$ designs (triplanes) and proved that if \mathcal{D} is a nontrivial symmetric $(v, k, 3)$ design with a flag-transitive and point-primitive automorphism group G , then \mathcal{D} has parameters $(11, 6, 3)$, $(15, 7, 3)$, $(45, 12, 3)$ or G is a subgroup of $AGL_1(q)$ where $q = p^m$ with $p \geq 5$ prime [13, 48, 49, 50, 51]. Nontrivial symmetric $(v, k, 4)$ designs admitting flag-transitive and point-primitive almost simple automorphism group whose socle is an alternating group or $PSL_2(q)$ have also been investigated [12, 52]. Note that for $\lambda \leq 3$, there is no flag-transitive and point-primitive nontrivial symmetric designs \mathcal{D} whose automorphisms group is an almost simple group with socle a finite simple exceptional group of Lie type [39, 41, 50]. As an immediate consequence of Theorem 1.1, by considering small possible values of q in Tables 1 and 2, the same result is true for $\lambda \leq 87$.

Corollary 1.3. *Let \mathcal{D} be a nontrivial symmetric (v, k, λ) design with $\lambda \leq 87$. Then there is no flag-transitive and point-primitive automorphisms group G of \mathcal{D} with the socle X a finite simple exceptional group of Lie type.*

In order to prove Theorem 1.1, we observe that, point-stabilisers H of a flag-transitive automorphisms group G must be large subgroups, that is to say, $|G| \leq |H|^3$. As G is also point-primitive, the subgroup H is also maximal in G , and by a reduction theorem of Liebeck and Seitz [29, Theorem 2] and the same arguments as in [3], we obtain large maximal subgroups of G in Theorem 3.1. Then we frequently apply Lemma 2.4 as a key tool and prove Theorem 1.1 in Section 4. We also use GAP [14] and apply Lemma 2.6 and Algorithm 1 for computational arguments.

In the case where G is imprimitive, Praeger and Zhou [40] studied point-imprimitive symmetric (v, k, λ) designs, and determined all such possible designs for $\lambda \leq 10$. This motivates Praeger and Reichard [23] to classify flag-transitive symmetric $(96, 20, 4)$ designs. As a result of their work, the only examples for flag-transitive, point-imprimitive symmetric $(v, k, 4)$ designs are $(15, 8, 4)$ and $(96, 20, 4)$ designs. In a recent study of imprimitive flag-transitive designs [5], Cameron and Praeger gave a construction of a family of designs with a specified point-partition, and determine the subgroup of automorphisms leaving invariant the point-partition. They gave necessary and sufficient conditions for a design in the family to possess a flag-transitive group of automorphisms preserving the specified point-partition. Consequently, they gave examples of flag-transitive designs in the family, including a new symmetric 2 -(1408, 336, 80) design with automorphism group $2^{12} : ((3 \cdot M_{22}) : 2)$, and a construction of one of the families of the symplectic designs exhibiting a flag-transitive, point-imprimitive automorphism group.

We here adopt the standard Lie notation for groups of Lie type, so for example we write $A_{n-1}^-(q)$ in place of $PSU_n(q)$, $D_n^-(q)$ instead of $P\Omega_{2n}^-(q)$, and $E_6^-(q)$ for ${}^2E_6(q)$. Also note that we may assume $q > 2$ if $G = G_2(q)$ since $G_2(2)' \cong PSU_3(3)$, and we

view the Tits group ${}^2F_4(2)'$ as a sporadic group. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leq \text{Aut}(X)$ where X is a nonabelian simple group. For a given positive integer n and a prime divisor p of n , we denote the p -part of n by n_p , that is to say, $n_p = p^t$ with $p^t \mid n$ but $p^{t+1} \nmid n$. We also denote the n -th cyclotomic polynomial by $\Phi_n(q)$. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [11, 15, 22].

2. PRELIMINARIES

In this section, we state some useful facts in both design theory and group theory. Recall that a group G is called almost simple if $X \trianglelefteq G \leq \text{Aut}(X)$, where X is a (nonabelian) simple group. If H is a maximal subgroup of an almost simple group G with socle X , then $G = HX$, and since we may identify X with $\text{Inn}(X)$, the group of inner automorphisms of X , we also conclude that $|H|$ divides $|\text{Out}(X)| \cdot |X \cap H|$. This implies the following elementary and useful fact:

Lemma 2.1. [2, Lemma 2.2] *Let G be an almost simple group with socle X , and let H be maximal in G not containing X . Then*

- (a) $G = HX$;
- (b) $|H|$ divides $|\text{Out}(X)| \cdot |X \cap H|$.

Lemma 2.2. *Suppose that \mathcal{D} is a symmetric (v, k, λ) design admitting a flag-transitive and point-primitive almost simple automorphism group G with socle X of Lie type in odd characteristic p . Suppose also that the point-stabiliser G_α , not containing X , is not a parabolic subgroup of G . Then $\gcd(p, v-1) = 1$.*

Proof. Note that G_α is maximal in G , then by Tits' Lemma [42, 1.6], p divides $|G : G_\alpha| = v$, and so $\gcd(p, v-1) = 1$. \square

Lemma 2.3. [26, 3.9] *If X is a group of Lie type in characteristic p , acting on the set of cosets of a maximal parabolic subgroup, and X is not $\text{PSL}(n, q)$, $P\Omega_{2m}^+(q)$ (with m odd), nor $E_6(q)$, then there is a unique subdegree which is a power of p .*

Note that a symmetric (v, k, λ) design of order $n := k - \lambda$ satisfies $4n - 1 \leq v \leq n^2 + n + 1$ and $\lambda(v-1) = k(k-1)$ with $\lambda < k < v-1$. This together with Lemma 2.4 and 2.6 below are helpful in our computational arguments.

Lemma 2.4. [1, Lemma 2.1] *Let \mathcal{D} be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of \mathcal{D} . If α is a point in \mathcal{V} and $H := G_\alpha$, then*

- (a) $k(k-1) = \lambda(v-1)$;
- (b) $4\lambda(v-1) + 1$ is square;
- (c) $k \mid |H|$ and $\lambda v < k^2$;
- (d) $k \mid \gcd(\lambda(v-1), |H|)$;
- (e) $k \mid \lambda d$, for all subdegrees d of G .

For a point stabiliser H of an automorphisms group G of a flag-transitive design \mathcal{D} , by Lemma 2.4(c), we conclude that $\lambda|G| \leq |H|^3$, and so we have that

Corollary 2.5. *Let \mathcal{D} be a flag-transitive (v, k, λ) symmetric design with automorphism group G . Then $|G| \leq |G_\alpha|^3$, where α is a point in \mathcal{D} .*

Lemma 2.6. *Let \mathcal{D} be a symmetric (v, k, λ) design, and let $mk = \lambda d$, where $d := \gcd(v-1, |\text{Out}(X)| \cdot |X \cap H|)$ for some positive integer m . Then the following properties hold:*

- (a) $m \mid k - 1$ and so $\gcd(m, k) = 1$;
- (b) $\lambda = \lambda_1 \lambda_2$, where $\lambda_1 = \gcd(\lambda, k - 1)$ and $\lambda_2 = \gcd(\lambda, k)$;
- (c) If $k_1 := (k - 1)/\lambda_1$ and $k_2 := k/\lambda_2$, then k_2 divides d and λ_1 divides m_1 .
Moreover,

$$\lambda_1 < k_2/2, \gcd(k_1, k_2) = 1 \text{ and } \gcd(\lambda_1, k_2) = 1.$$

Proof. (a) Since $mk = \lambda d \mid \lambda(v - 1)$ and $\lambda(v - 1) = k(k - 1)$, it follows that mk divides $k(k - 1)$, and so m is a divisor of $k - 1$, and hence it is coprime to k .

(b) This part follows immediately from part (a) and $k(k - 1) = \lambda(v - 1)$.

(c) Note that λ_1 is relatively prime to k_2 . Since $mk = \lambda d$, it follows that $k_2 m = \lambda_1 d$, and so k_2 divides d and λ_1 is a divisor of m . Since also $\lambda < k$ and $mk = \lambda d$, we conclude that $m < d$, moreover $k_2 \neq 1$. Note that $k \leq v/2$. Then $\lambda_1 = k(k - 1)/\lambda_2(v - 1) \leq k_2(v - 2)/2(v - 1) < k_2/2$. The rest is obvious. \square

The following algorithm written based on Lemmas 2.4 and 2.6 sometimes helps to rule out the numerical cases. The input of Algorithm 1 is a list of possible (q, a) , see for example the last column of Tables 1 and 2. The output is possible parameters (v, k, λ) for desired symmetric designs.

Algorithm 1. An algorithm based on Lemmas 2.4 and 2.6

Data: A list L_1 of pairs (q, a)

Result: Possible parameters (q, a, n, v, k, λ)

initialization;

for (q, a) **in** L_1 **do**

$v := |G : H|$;

$d := \text{DivisorsInt}(d) \setminus \{1\}$;

for k_2 **in** d **do**

$k_1 := (v - 1)/k_2$;

if $\gcd(k_2, k_1) = 1$ **then**

$\lambda_1 := 0$;

while $\lambda_1 \leq k_2/2$ **do**

$\lambda_1 := \lambda_1 + 1$;

if $\gcd(\lambda_1, k_2) = 1$ **then**

$k := 1 + (\lambda_1 * k_1)$;

$\lambda_2 := k/k_2$;

if $d * \lambda_2 \bmod k = 0$ and $k \bmod k_2 = 0$ and $k \leq v/2$ **then**

$\lambda := \lambda_1 * k/k_2$;

if $\text{IsInt}(\lambda)$ and $\lambda < k$ **then**

$n := k - \lambda$;

if $4 * n - 1 \leq v$ and $v \leq n^2 + n + 1$ **then**

Print (q, a, n, v, k, λ) ;

end

end

end

end

end

end

end

end

TABLE 3. Large maximal non-parabolic subgroups of almost simple groups G with socle X exceptional finite simple groups.

X	$X \cap H$ or type of H	Conditions
${}^2B_2(q)$ ($q = 2^{2n+1} \geq 8$)	$(q + \sqrt{2q} + 1):4$ ${}^2B_2(q^{1/3})$	$q = 8, 32$ $q > 8$
${}^2G_2(q)$ ($q = 3^{2n+1} \geq 27$)	$A_1(q)$ ${}^2G_2(q^{1/3})$	
${}^3D_4(q)$	$A_1(q^3)A_1(q), (q^2 + \epsilon 1q + 1)A_2^\epsilon(q), G_2(q),$ ${}^3D_4(q^{1/2})$ $7^2 : SL_2(3)$	$\epsilon = \pm$ $q = 2$
${}^2F_4(q)$ ($q = 2^{2n+1} \geq 8$)	${}^2B_2(q)^2, B_2(q), {}^2F_4(q^{1/3})$ $SU_3(q) : 2, PGU_3(q) : 2$ $L_3(3):2, L_2(25), A_6 \cdot 2^2, 5^2:4A_4$	$q = 8$ $q = 2$
$G_2(q)$	$A_2^\epsilon(q), A_1(q)^2, G_2(q^r)$ ${}^2G_2(q)$ $G_2(2)$ $L_2(13), J_2$ J_1 $2^3.L_3(2)$	$r = 2, 3$ $q = 3^a, a \text{ is odd}$ $q = 5, 7$ $q = 4$ $q = 11$ $q = 3, 5$
$F_4(q)$	$B_4(q), D_4(q), {}^3D_4(q)$ $F_4(q^{1/r})$ $A_1(q)C_3(q)$ $C_4(q), C_2(q^2), C_2(q)^2$ ${}^2F_4(q)$ ${}^3D_4(2)$ $Alt_{9-10}, A_3(3), J_2$ $A_1(q)G_2(q)$ $Sym_6 \wr Sym_2, F_4(2)$	$r = 2, 3$ $p \neq 2$ $p = 2$ $q = 2^{2n+1} \geq 2$ $q = 3$ $q = 2$ $q > 3 \text{ odd}$ $q = 2$
$E_6^\epsilon(q)$	$A_1(q)A_5^\epsilon(q), F_4(q)$ $(q - \epsilon)D_5^\epsilon(q)$ $C_4(q)$ $E_6^\pm(q^{1/2})$ $E_6^\epsilon(q^{1/3})$ $(q - \epsilon)^2.D_4(q)$ $(q^2 + \epsilon q + 1).{}^3D_4(q)$ $J_3, Alt_{12}, B_3(3), Fi_{22}$	$\epsilon = -$ $p \neq 2$ $\epsilon = +$ $(\epsilon, q) \neq (+, 2)$ $(\epsilon, q) \neq (-, 2)$ $(\epsilon, q) = (-, 2)$
$E_7(q)$	$(q - \epsilon)E_6^\epsilon(q), A_1(q)D_6(q), A_7^\epsilon(q),$ $A_1(q)F_4(q), E_7(q^{1/r})$ Fi_{22}	$\epsilon = \pm \text{ and } r = 2, 3$ $q = 2$
$E_8(q)$	$A_1(q)E_7(q), D_8(q), A_2^\epsilon(q)E_6^\epsilon(q), E_8(q^{1/r})$	$\epsilon = \pm \text{ and } r = 2, 3$

3. LARGE MAXIMAL SUBGROUPS OF EXCEPTIONAL ALMOST SIMPLE GROUPS

Recall that a proper subgroup H of G is said to be large if the order of H satisfies the bound $|G| \leq |H|^3$. In this section, we prove Theorem 3.1 below. Here we apply the same method as in [3].

Theorem 3.1. *Let G be a finite almost simple group whose socle X is a finite exceptional simple group, and let H be a maximal non-parabolic subgroup of G not containing X . If H is a large subgroup of G , then H is one of the subgroups listed in Table 3.*

We will assume G is a finite almost simple group with socle an exceptional Lie type. Note that the order of G is given in [18, Table 5.1.B]. We first observe the following elementary lemma:

Lemma 3.2. *Let G be a finite almost simple group with socle non-abelian simple group X , and let H be a maximal subgroup of G not containing X . Then H is a large subgroup of G if and only if $|X| < b^2|X \cap H|^3$, where $b = |G|/|X|$ divides $|\text{Out}(X)|$.*

Proof. Let H be a maximal large subgroup of G and $b = |G|/|X|$. Since $|H| = b \cdot |X \cap H|$ and $|G| = b \cdot |X|$, $|X| \leq b^2|X \cap H|^3$. Conversely, let $b = |G|/|X|$ and $|X| < b^2|X \cap H|^3$. Note that $|X| = |G|/b$ and $|H| = b \cdot |X \cap H|$. Thus H is a large subgroup of G . \square

Remark 3.3. By Lemma 3.2, to determine the large maximal subgroups H of G , we need to verify $|X| < b^2|X \cap H|^3$ with $b \mid |\text{Out}(X)|$. It is worthy to note that such subgroups satisfying $|X| \leq |H \cap X|^3$ have been determined in [3, Theorem 7], and so in what follows we only need to find large maximal subgroups H of G satisfying

$$|H \cap X|^3 < |X| \leq b^2|H \cap X|^3, \quad (3.1)$$

where $b \mid |\text{Out}(X)|$.

Proposition 3.4. *The conclusion to Theorem 3.1 holds when X is one of the groups $E_6^\epsilon(2)$, $F_4(2)$, ${}^2F_4(q)$, ${}^3D_4(q)$, $G_2(q)$, ${}^2G_2(q)$, ${}^2B_2(q)$.*

Proof. In each of these cases, the maximal subgroups of G have been determined and the relevant references are listed below (also see [47, Chapter 4]). Note that the list of maximal subgroups of ${}^2E_6(2)$ presented in the Atlas [8] is complete (see [16, p.304]).

G	$E_6^\epsilon(2)$	$F_4(2)$	${}^2F_4(q)$	${}^3D_4(q)$	$G_2(q)$	${}^2G_2(q)$	${}^2B_2(q)$
Ref.	[8, 21]	[34]	[33]	[20]	[9, 19]	[19, p. 61]	[43]

Now it is straightforward to verify Theorem 3.1 for these groups. In particular, we note that every maximal parabolic subgroup of G is large. \square

Let us now turn our attention to the remaining cases:

$$F_4(q), E_6^\epsilon(q), E_7(q), E_8(q),$$

where $\epsilon = \pm 1$, $q = p^a$ and p is a prime (and $q > 2$ if $X = E_6^\epsilon(q)$ or $F_4(q)$).

Let \mathfrak{G} be a simple adjoint algebraic group of exceptional type over an algebraically closed field K of characteristic p , and if $p > 0$ let σ be a surjective endomorphism of \mathfrak{G} such that $X = O^{p'}(\mathfrak{G}_\sigma)$ is a finite simple group of Lie type. Let G be a finite almost simple group with socle X , where X is an exceptional group of Lie type and let H be a maximal subgroup of G , not containing X . Let also $H_0 := \text{Soc}(X \cap H)$. Denote by Alt_n and Sym_n , the alternating and symmetric groups of degree n , respectively. We will apply the following reduction theorem of Liebeck and Seitz, see [29, Theorem 2].

Theorem 3.5. *Let $X = O^{p'}(\mathfrak{G}_\sigma)$ be a finite exceptional group of Lie type, let G be a group such that $X \leq G \leq \text{Aut}(X)$, and let H be a maximal non-parabolic subgroup of G . Then one of the following holds:*

- (i) $H = N_G(\bar{M}_\sigma)$, where \bar{M} is a σ -stable closed subgroup of positive dimension in G . The possibilities are obtained in [27, 31].
 - (a) H is reductive of maximal rank (as listed in Table 5.1 in [27], see also [31]).
 - (b) $\mathfrak{G} = E_7$, $p > 2$ and $H = (2^2 \times P\Omega_8(q) \cdot 2^2) \cdot \text{Sym}_3$ or ${}^3D_4(q).3$.

- (c) $\mathfrak{G} = E_8$, $p > 5$ and $H = PGL_2(q) \times \text{Sym}_5$.
- (d) $H = \bar{M}_\sigma$ with $H_0 = \text{Soc}(H)$ as in Table 4 below.
- (ii) H is an exotic local subgroup recorded in [7, Table 1].
- (iii) $\mathfrak{G} = E_8$, $p > 5$ and $H = (\text{Alt}_5 \times \text{Alt}_6).2^2$.
- (iv) H is of the same type as G over a subfield of F_q of prime index.
- (v) H is almost simple, and not of type (i) or (iv).

TABLE 4. Possibilities for H_0 in Theorem 3.5(i)(d).

X	H_0
$F_4(q)$	$A_1(q) \times G_2(q)$ ($p > 2, q > 3$)
$E_6^\epsilon(q)$	$A_2(q) \times G_2(q)$, $A_2^-(q) \times G_2(q)$ ($q > 2$)
$E_7(q)$	$A_1(q) \times A_1(q)$ ($p > 3$), $A_1(q) \times G_2(q)$ ($p > 2, q > 3$), $A_1(q) \times F_4(q)$ ($q > 3$), $G_2(q) \times PSp_6(q)$
$E_8(q)$	$A_1(q) \times A_1^\epsilon(q)$ ($p > 3$), $A_1(q) \times G_2(q) \times G_2(q)$ ($p > 2, q > 3$) $G_2(q) \times F_4(q)$ ($q > 3$), $A_1(q) \times G_2(q^2)$ ($p > 2, q > 3$)

Remark 3.6. Suppose H is almost simple with socle H_0 , as in part (v) of Theorem 3.5. Then

- (a) If $H_0 \notin \text{Lie}(p)$ then the possibilities for H_0 have been determined up to isomorphism, see [30, Tables 10.1–10.4].
- (b) If $H_0 \in \text{Lie}(p)$ and $\text{rk}(H_0) > \frac{1}{2}\text{rk}(G)$. Then by applying [28, Theorem 3], $G = E_6^\epsilon(q)$ and $H_0 = C_4(q)$ (q odd) or $H_0 = F_4(q)$.
- (c) If $H_0 \in \text{Lie}(p)$ and $\text{rk}(H_0) \leq \frac{1}{2}\text{rk}(G)$, then $|H| \leq 12aq^{56}$, $4aq^{30}$, $4aq^{28}$ or $4aq^{20}$ accordingly as $G = E_8$, E_7 , E_6^ϵ or F_4 , see [32, Theorem 1.2]. Moreover, if H_0 is defined over \mathbb{F}_s for some p -power s , then one of the following holds (see [24, Theorem 2] for the values of $u(G)$ in part (3)):
 - (1) $s \leq 9$;
 - (2) $H_0 = A_2^\epsilon(16)$;
 - (3) $H_0 \in \{A_1(s), {}^2B_2(s), {}^2G_2(s)\}$ and $s \leq (2, p-1) \cdot u(G)$, where $u(G)$ is defined as follows:

G	G_2	F_4	E_6	E_7	E_8
$u(G)$	12	68	124	388	1312

Note that if $X \cap H$ is a parabolic subgroup, then it is easy to check that (3.1) holds, so for the remainder, we will assume that $X \cap H$ is non-parabolic.

Proposition 3.7. Suppose $X = F_4(q)$ with $q = p^a > 2$. If H is large, then H is of type $B_4(q)$, $D_4(q)$, ${}^3D_4(q)$, $A_1(q)C_3(q)$ ($p \neq 2$), $C_4(q)$ ($p = 2$), $C_2(q)^2$, $C_2(q^2)$ ($p = 2$), $A_1(q)G_2(q)$, ${}^2F_4(q)$ ($q = 2^a$, a odd), $F_4(q_0)$ with $q = q_0^r$ for $r = 2, 3$, or ${}^3D_4(2)$ ($q = 3$).

Proof. By Theorem 3.5, H is of one of the types (i)–(v). We note that H is non-large if $|H| < q^{16}$. So we restrict our attention to the case where $|H| \geq q^{16}$. Also recall that here we need only to deal with the subgroups satisfying (3.1).

Suppose H is of type (i). Here \bar{M} is listed in [31]. If H is of maximal rank, then the possibilities for H can be read off from [27, Table 5.1]. It is now straightforward to check that the only possibilities for H is $A_1(q)C_3(q)$ with q odd, $C_4(q)$, $C_2(q)^2$ and $C_2(q^2)$ with $p = 2$, $B_4(q)$, $D_4(q)$ and ${}^3D_4(q)$. If the socle H_0 of H is $A_1(q)G_2(q)$ with $q > 3$ odd (see Table 4), then by [3, Theorem 5], we must have $G \neq X$.

Clearly H is not of type (iii). Suppose now H is of type (ii). Then H is too small to be large. Let now H is of type (iv), then (3.1) holds only for $F_4(q^{\frac{1}{r}})$ with $r = 1, 2$. Note that the latter case may occur when $G \neq X$.

Suppose H is of type (v). Then H is almost simple but not of type (i) and (iv). Let H_0 denote the socle of H . If $H_0 \notin \text{Lie}(p)$ then the possibilities for H are recorded in [30, Tables 10.1–10.4], and it is easy to check that no large examples arise if $q > 3$. However, if $q = 3$ then $H_0 = {}^3D_4(2)$ is a possibility for $X = F_4(3)$. Now assume $H_0 \in \text{Lie}(p)$ and $r = \text{rk}(H_0) \leq 2$. Here [32, Theorem 1.2] gives $|H| < 4aq^{20}$, so some additional work is required. There are several cases to consider.

Write $q = p^a$ and $H_0 = X_r^\epsilon(s)$, where $s = p^b$. First assume $H_0 = A_2(s)$, so $q^{16} \leq |H| < s^{10}$ and thus $b/a \geq 16/10$. By considering the primitive prime divisor p_{3b} of $|H|$ we deduce that $b/a \in \{4, 2, 8/3\}$. The case $b/a = 4$ is ruled out in the proof of [32, Theorem 1.2], and Remark 3.6(c) rules out the case $b/a = 8/3$. Therefore $H_0 = A_2(q^2)$ is the only possibility, and we calculate that $|H|^3 < |G|$ unless $q = 4$ and $H = \text{Aut}(A_2(16))$. Similarly, if $H_0 = A_2^-(s)$ then H is large if and only if $q = 4$ and $H = \text{Aut}(A_2^-(16))$. However, $A_2^-(16)$ is not a subgroup of $F_4(4)$, see proof of [3, Lemma 5.7].

Next assume $H_0 = C_2(s)$. Here $b/a \geq 17/11$ since $|H| < s^{11}$, and by considering p_{4b} we deduce that $b/a \in \{2, 3\}$. The case $b/a = 3$ is eliminated in the proof of [32, Theorem 1.2], so we can assume $H_0 = C_2(q^2)$. As noted in Remark 3.6(ii), such a subgroup is non-maximal if $q > 3$, so let us assume $q = 3$. Note by [3, Lemma 5.7] that $H_0 = C_2(9)$ is not a subgroup of $G = F_4(3)$. The case $H_0 = B_2(s)$ is entirely similar. Finally, the remaining possibilities for H_0 can be ruled out in the usual manner. \square

Proposition 3.8. *Let H be a maximal non-parabolic subgroup of $G = E_6^\epsilon(q)$, where $q > 2$. Then H is large if and only if H is of type $(q - \epsilon)D_5^\epsilon(q)$ ($\epsilon = -$), $A_1(q)A_5^\epsilon(q)$, $F_4(q)$, $(q - \epsilon)^2.D_4(q)$, $(q^2 + \epsilon q + 1).{}^3D_4(q)$, $C_4(q)$ ($p \neq 2$), $E_6^\epsilon(q_0)$ with $q = q_0^r$ for $(\epsilon, \epsilon, r) = (+, +, 2), (+, +, 3), (+, -, 2), (-, -, 3)$.*

Proof. If $|H| \leq q^{24}$, then $|H|^3 < |G|$, so in this case we may assume that $|H| > q^{24}$. We now apply Theorem 3.5.

If H is of type (i), then $H = N_G(\bar{M}_\sigma)$, where $\bar{M} = T_1D_5, A_1A_5, T_2D_4.S_3$. If H is of type (ii) then $H = 3^6.\text{SL}_3(3)$ (with $p \geq 5$) is the only possibility, and H is non-large. Case (iii) does not apply here. Now assume H is a subfield subgroup of type $E_6^\epsilon(q_0)$ with $q = q_0^r$, then it is easy to see that $q = q_0^r$ for $(\epsilon, \epsilon, r) = (+, +, 2), (+, +, 3), (+, -, 2), (-, -, 3)$.

Finally, let us assume H is almost simple and not of type (i) or (iv). Let H_0 denote the socle of H . First assume $H_0 \notin \text{Lie}(p)$. Here the possibilities for H_0 can be read off from [30, Tables 10.1–10.4] and it is straightforward to check that no large subgroups of this type arise. If $H_0 \in \text{Lie}(p)$ and $\text{rk}(H_0) \geq 3$, the subgroups of type F_4 and C_4 ($p \neq 2$) are large. Therefore, to complete the proof of the lemma we may assume that $H_0 \in \text{Lie}(p)$ and $\text{rk}(H_0) \leq 3$. Here [32, Theorem 1.2(iii)] gives $|H| \leq 4q^{28} \log_p q$, so some additional work is required.

We proceed as in the proof of [32, Theorem 1.2], using the method described in [25, Step 3, p.310]. Write $q = p^a$ and $H_0 = X_r^{\epsilon'}(s)$, where $r = \text{rk}(H_0)$ and $s = p^b$. We consider the various possibilities for X_r (with $r \leq 3$) in turn. Recall that if $c \geq 2$ and $d \geq 3$ are integers (and $(c, d) \neq (2, 6)$), then c_d denotes the largest primitive prime divisor of $c^d - 1$. Here we may assume by [4, Lemma .13] that $|H| \leq q^{32}$.

To illustrate the general approach, consider the case $H_0 = A_3(s)$ with $\epsilon = +$. Here $q^{24} < |H| \leq |\text{Aut}(H_0)| < s^{17}$, and thus $b/a \geq 25/17$. Now p_{4b} divides $|H|$, and thus $|G|$, so $4b$ divides one of the numbers $6a, 8a, 9a, 12a$, whence $b/a \in \{3, 9/4, 2, 3/2\}$. Moreover, since p_{3b} divides $|G|$ (note that $(p, b) \neq (2, 2)$ since we are assuming that $q > 2$) we deduce that $b/a \in \{3, 2, 3/2\}$. However, $H_0 \neq A_3(q^2)$ by the proof of [32, Theorem 1.2], and we have $|H| < q^{24}$ if $H_0 = A_3(q^{3/2})$, and $|H| > q^{32}$ if $H_0 = A_3(q^3)$. This eliminates the case where $H_0 = A_3(s)$. By the same manner, the other cases does not give rise to any large subgroups, see also proof of [3, Lemma 5.6]. \square

Proposition 3.9. *Let H be a maximal non-parabolic subgroup of $G = E_7(q)$. Then H is large if and only if H is of type $(q - \epsilon)E_6^\epsilon(q)$, $A_1(q)D_6(q)$, $A_7^\epsilon(q)$, $A_1(q)F_4(q)$, $E_7(q_0)$ with $q = q_0^r$ for $r = 2, 3$, or Fi_{22} for $q = 2$.*

Proof. We proceed as in the proof of the previous proposition. If $|H| < q^{43}$, then $|H|^3 < |G|$, so to complete the analysis of this case we may assume that $|H| \geq q^{43}$. We now apply Theorem 3.5.

By inspecting [31] and [7, Table 1], it is easy to check that the only examples of type (i) are $N_G(\bar{M}_\sigma)$, with $\bar{M} = T_1E_6.2$, $A_1D_6.2$, $A_7.2$ or A_1F_4 . Next suppose H is a subfield subgroup of type $E_7(q_0)$ with $q = q_0^r$. If $r \geq 5$, then H is non-large. The cases (ii) and (iii) do not arise.

To complete the analysis, let us assume H is almost simple, and not of type (i) or (iv). Let H_0 denote the socle of H . If $H_0 \notin \text{Lie}(p)$ then by inspecting [30, Tables 10.1–10.4], we deduce that H_0 can be Fi_{22} for $q = 2$. Finally, we may assume $H_0 \in \text{Lie}(p)$ and $\text{rk}(H_0) \leq 3$ (see Remark 3.6(ii)). Here [32, Theorem 1.2(ii)] states that $|H| < 4q^{30} \log_p q$, and thus H is non-large. \square

Proposition 3.10. *Let H be a maximal non-parabolic subgroup of $G = E_8(q)$. Then H is large if and only if H is of type $A_1(q)E_7(q)$, $D_8(q)$, $A_2^\epsilon(q)E_6^\epsilon(q)$ or $E_8(q_0)$ with $q = q_0^r$ for $r = 2, 3$.*

Proof. Clearly, if $|H| \leq q^{81}$, then H is non-large, so it remains to consider the maximal subgroups H that satisfy the bounds $|H| > q^{81}$. By Theorem 3.5, H is of type (i)–(v).

If H is of type (i) of maximal rank, then by [27, Table 5.1], the only possibilities for H are D_8 , $A_1(q)E_7(q)$ and $A_2^\epsilon(q)E_6^\epsilon(q)$. For other cases in type (i), $|H|$ in the desired range (3.1). The possibilities in (ii) are recorded in [7, Table 1]; either $|H| = 2^{15}|\text{SL}_5(2)|$, or $|H| = 5^3|\text{SL}_3(5)|$ (both with q odd). In both cases, H is non-large. Clearly, we can eliminate subgroups of type (iii), and a straightforward calculation shows that a subfield subgroup $H = E_8(q_0)$ with $q = q_0^r$, r prime is large only if $r = 2, 3$.

Finally, let us assume H is almost simple, and not of type (i) or (iv). Let H_0 denote the socle of H , and recall that $\text{Lie}(p)$ is the set of finite simple groups of Lie type in characteristic p . First assume that $H_0 \notin \text{Lie}(p)$, in which case the possibilities for H_0 are listed in [30, Tables 10.1–10.4]. If H_0 is an alternating or sporadic group then $|H| \leq |\text{Th}|$ and thus H is non-large. Similarly, if H is a group of Lie type then we get $|H| \leq |\text{PGL}_4(5)|2$, and again we deduce that H is non-large. Finally, suppose $H_0 \in \text{Lie}(p)$. By Remark 3.6(c) we have $\text{rk}(H_0) \leq 4$, so $|H| < 12q^{56} \log_p q$ by [32, Theorem 1.2(i)], and thus H is non-large. \square

4. PROOF OF THE MAIN RESULT

In this section, suppose that \mathcal{D} is a nontrivial (v, k, λ) symmetric design and G is an almost simple automorphism group G with simple socle X , where X is a finite simple exceptional group of Lie type. Let now G be a flag-transitive and point-primitive automorphism group of \mathcal{D} . Then by corollary 2.5, $H := G_\alpha$ is a large subgroups of G , where α is a point of \mathcal{D} . So $X \cap H$ is (isomorphic to) one of the subgroups in Theorem 3.1. Moreover, by Lemma 2.1,

$$v = \frac{|X|}{|X \cap H|}. \quad (4.1)$$

4.1. Parabolic case. In this section we deal with the case where $H := G_\alpha$ is a maximal parabolic subgroup of G .

Proposition 4.1. *If $X \cap H$ is a parabolic subgroup of X , then $X = G_2(q)$ and $X \cap H = [q^5] : \text{GL}_2(q)$ and \mathcal{D} is a symmetric $(q^5 + q^4 + q^3 + q^2 + q + 1, q^5, q^5 - q^4)$ design.*

Proof. For pairs $(X, X \cap H)$ as in the first column of Table 5, we observe by (4.1), v is as in the forth column of Table 5. Let now $v - 1 = |v - 1|_p \cdot s(q)$ for some polynomial $s(q)$. Then $|v - 1|_p = q^t$ is given in the fifth column of the Table 5. By Lemma 2.3 and [41, Section 8, p. 345] there is a unique subdegree p^b . Then k must divide $\lambda \gcd(v - 1, p^b)$, and since $s(q)$ is coprime to p , it follows that k divides λq^t , where q^t is as in the fifth column of the Table 5.

Let first $(X, X \cap H)$ be as in line 1 or 2 of Table 5. Then $v - 1 = q^t$, where (p, t) is $(2, 2)$ or $(3, 3)$, respectively. Note by Lemma 2.4(a) that $k(k - 1) = \lambda q^t$. If k is coprime to p , then since $k \mid \lambda q^t$, we conclude that $k \leq \lambda$, which is impossible. If $\gcd(k, p) = p$, then $k - 1$ divides λ , which is also impossible.

For the remaining cases, as k divides λq^t , let m be a positive integer such that $mk = \lambda q^t$. Since $\lambda < k$, we have that

$$m < q^t, \quad (4.2)$$

where q^t is as in the fifth column of the Table 5. Note by Lemma 2.4(a) that $k(k - 1) = \lambda(v - 1)$. So by the fact that $mk = \lambda q^t$ and $v - 1 = q^t s(q)$, we have that

$$k = ms(q) + 1 \quad \text{and} \quad \lambda = m^2 t(q) + \frac{m^2 r(q) + m}{q^t}, \quad (4.3)$$

for some polynomials $r(q)$ and $t(q)$. For convenience, we give $r(q)$ in the sixth column of Table 5. We now discuss each case separately.

Let $(X, X \cap H)$ be as in line 3, 5 or 8, ..., 33 of Table 5. Then $|v - 1|_p = q$ and $r(q) = 1$. Thus q must divide $m^2 + m$. Note that q is prime power. Then by (4.2), we conclude that $m = q - 1$. Now (4.3) implies that $k = (q - 1)s(q) + 1$. Set $h(q) := \gcd(k, |X \cap H|)$. For each X and $X \cap H$, by calculation, $h(q)$ is as in the last column of Table 5. By Lemmas 2.4(c) and 2.1(b), k divides $|\text{Out}(X)||X \cap H|$, and so we conclude that $k/h(q)$ divides $|\text{Out}(X)|$. In each case, we observe that $k/h(q) > q^4$, then $|\text{Out}(X)| > q^4$, which is impossible.

Let $(X, X \cap H)$ be as in line 4 of Table 5. Then $|v - 1|_p = q^3$ and $r(q) = q + 1$. Moreover, by (4.3), we have

$$\lambda = m^2(q^5 + q^2 + q) + \frac{m^2(q + 1) + m}{q^3}$$

This implies that $q^3 \mid m^2(q+1) + m$, and so by (4.2), q^3 divides $m(q+1) + 1$. Let c be a positive integer such that $m(q+1) + 1 = cq^3$. Then

$$m = \frac{cq^3 - 1}{q+1} = cq(q-1) + \frac{cq-1}{q+1}.$$

Therefore $q+1$ must divide $cq-1$. Note by (4.2) that $c \leq q$. Set $b := q - c$. Then $q+1$ must divide $cq-1 = q^2 - bq - 1 = (q^2 - 1) - bq$, and so $q+1$ divides bq , and hence $q+1$ must divide b , and this contradicts the fact that $b \leq q-2$.

Let $(X, X \cap H)$ be as in line 6 of Table 5. Then $|v-1|_p = q^2$ and $r(q) = q+1$, and by (4.3), we have

$$\lambda = m^2(q^7 + q^5 + q^4 + q^2 + q) + \frac{m^2(q+1) + m}{q^2}.$$

This implies that $q^2 \mid m^2(q+1) + m$. Now (4.2) yields $q^2 \mid m(q+1) + 1$. So there exists a positive integer $c < q+1$ such that

$$m = \frac{cq^2 - 1}{q+1} = c(q-1) + \frac{c-1}{q+1}.$$

Thus $q+1$ must divide $c-1$, which is a contradiction.

Let $(X, X \cap H)$ be as in line 7 of Table 5. Here $v = q^5 + q^4 + q^3 + q^2 + q + 1$, and so $|v-1|_p = q$, and by (4.3), we have

$$k = m(q^4 + q^3 + q^2 + q + 1) + 1 \quad \text{and} \quad \lambda = m^2(q^3 + q^2 + q + 1) + \frac{m^2 + m}{q}. \quad (4.4)$$

So by (4.2), we conclude that $m = q-1$. Therefore, (4.4) implies that $k = q^5$ and $\lambda = q^5 - q^4$. \square

4.2. Subfield and some numerical cases. In this section, we deal with the case where $H := G_\alpha$ is a maximal subfield subgroup of G and subgroups $X \cap H$ in Table 6 with q small.

Proposition 4.2. *If X and $X \cap H$ are as in Table 6, then there is no symmetric design with G as a flag-transitive and point-primitive automorphism group.*

Proof. If X and $X \cap H$ as in Table 6, then by (4.1) and Lemma 2.4, the parameters v and k are as in third and forth columns of Table 6, respectively. For each value of v and k , the equality $k(k-1) = \lambda(v-1)$ does not hold for any positive integer λ . \square

Proposition 4.3. *If $X \cap H$ is a subfield subgroup of X , then there is no symmetric design with G as a flag-transitive and point-primitive automorphism group.*

Proof. Let X and $X \cap H$ be as in the second and third columns of Table 7, respectively. Then by (4.1), we obtain v for the corresponding X and $X \cap H$ as in the forth column of Table 7. By Lemmas 2.4(d) and 2.1(b), the parameter k divides both $|\text{Out}(X)||X \cap H|$ and $\lambda(v-1)$. Note by Lemma 2.2 that $\gcd(v-1, p) = 1$. Thus

$$k \mid \lambda f(q_0) |\text{Out}(X)|,$$

where $f(q_0)$ is as in fifth column of Table 7. Therefore

$$k \mid \lambda f(q_0) |\text{Out}(X)|. \quad (4.5)$$

TABLE 5. Parameters for Proposition 4.1

Line	X	Type of $X \cap H$	v	$ v-1 _p$	$r(q)$	$h(q)$
1	${}^2B_2(q)$	$E_q \cdot E_q \cdot C_{q-1}$	$\Phi_4(q)$	q^2	na	na
2	${}^2G_2(q)$	$q^{1+1+1} : C_{q-1}$	$\Phi_2(q)\Phi_6(q)$	q^3	na	na
3	${}^3D_4(q)$	$q^{1+8} : SL_2(q^3) \cdot C_{q-1}$	$\Phi_2(q)\Phi_3(q)\Phi_6(q)\Phi_{12}(q)$	q	1	q
4	${}^3D_4(q)$	$q^{2+3+6} : SL_2(q) \cdot C_{q^3-1}$	$\Phi_2(q)\Phi_3(q)\Phi_6^2(q)\Phi_{12}(q)$	q^3	$q+1$	q^2
5	${}^2F_4(q)$	$[q^{10}] : {}^2B_2(q)C_{q-1}$	$\Phi_2^2(q)\Phi_4(q)\Phi_6(q)\Phi_{12}(q)$	q	1	q
6	${}^2F_4(q)$	$[q^{11}] : GL_2(q)$	$\Phi_2(q)\Phi_4^2(q)\Phi_6(q)\Phi_{12}(q)$	q^2	$q+1$	q^2
7	$G_2(q)$	$[q^5] : GL_2(q)$	$\Phi_1(q)\Phi_2(q)\Phi_3(q)\Phi_6(q)$	q	1	q^5
8	$F_4(q)$	$C_3(q)$	$\Phi_2(q)\Phi_3(q)\Phi_4(q)\Phi_6(q)\Phi_8(q)\Phi_{12}(q)$	q	1	q^3
9	$F_4(q)$	$B_3(q)$	$\Phi_2(q)\Phi_3(q)\Phi_4(q)\Phi_6(q)\Phi_8(q)\Phi_{12}(q)$	q	1	q^3
10	$F_4(q)$	$A_1(q)A_2(q)$	$\Phi_2^2(q)\Phi_3(q)\Phi_4^2(q)\Phi_6^2(q)\Phi_8(q)\Phi_{12}(q)$	q	1	q
11	$E_6^-(q)$	$A_5^-(q)$	$\Phi_2(q)\Phi_3(q)\Phi_4(q)\Phi_6(q)\Phi_8(q)\Phi_{12}(q)\Phi_{18}(q)$	q	1	q^2
12	$E_6^-(q)$	$O_8^-(q)$	$\Phi_2^3(q)\Phi_3(q)\Phi_4(q)\Phi_6^2(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)$	q	1	q^4
13	$E_6^-(q)$	$A_1(q)A_2(q^2)$	$\Phi_2^3(q)\Phi_3(q)\Phi_4(q)\Phi_6^2(q)\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)$	q	1	q^3
14	$E_6^-(q)$	$A_1(q^2)A_2(q)$	$\Phi_{18}(q)$ $\Phi_2^4(q)\Phi_3(q)\Phi_4(q)\Phi_6^3(q)\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)$	q	1	q^2
15	$E_6(q)$	$D_5(q)$	$\Phi_{18}(q)$ $\Phi_3^2(q)\Phi_6(q)\Phi_9(q)\Phi_{12}(q)$	q	1	q^3
16	$E_6(q)$	$A_1(q)A_4(q)$	$\Phi_2(q)\Phi_3^2(q)\Phi_4(q)\Phi_6^2(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$	q	1	q
17	$E_6(q)$	$A_1(q)A_2(q)A_2(q)$	$\Phi_1^{-1}(q)\Phi_2^2(q)\Phi_3(q)\Phi_4^2(q)\Phi_6^2(q)\Phi_8(q)\Phi_9(q)$ $\Phi_{10}(q)\Phi_{12}(q)$	q	1	q
18	$E_6(q)$	A_5	$\Phi_2(q)\Phi_3(q)\Phi_4(q)\Phi_6(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$	q	1	q^2
19	$E_7(q)$	$D_6(q)$	$\Phi_2(q)\Phi_3(q)\Phi_6(q)\Phi_7(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)$ $\Phi_{18}(q)$	q	1	q^3
20	$E_7(q)$	$A_1(q)A_5(q)$	$\Phi_1(q)\Phi_4^2(q)\Phi_3(q)\Phi_4(q)\Phi_6^2(q)\Phi_7(q)\Phi_8(q)$ $\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	q	1	q
21	$E_7(q)$	$A_6(q)$	$\Phi_1^{-1}(q)\Phi_2^3(q)\Phi_6(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{14}(q)$ $\Phi_{18}(q)$	q	1	q^2
22	$E_7(q)$	$A_1(q)A_2(q)A_3(q)$	$\Phi_2^3(q)\Phi_3(q)\Phi_4(q)\Phi_5(q)\Phi_6^3(q)\Phi_7(q)\Phi_8(q)$ $\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	q	1	q
23	$E_7(q)$	$A_2(q)A_4(q)$	$\Phi_2^4(q)\Phi_3(q)\Phi_4(q)\Phi_6^3(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)$ $\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	q	1	q
24	$E_7(q)$	$A_1(q)D_5(q)$	$\Phi_2^2(q)\Phi_3^2(q)\Phi_6^2(q)\Phi_7(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)$ $\Phi_{14}(q)\Phi_{18}(q)$	q	1	q
25	$E_7(q)$	$E_6(q)$	$\Phi_2^3(q)\Phi_6(q)\Phi_7(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{18}(q)$	q	1	q^4
26	$E_8(q)$	$D_7(q)$	$\Phi_2^2(q)\Phi_3^2(q)\Phi_4(q)\Phi_5(q)\Phi_6^2(q)\Phi_8(q)\Phi_9(q)$ $\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)\Phi_{20}(q)$ $\Phi_{24}(q)\Phi_{30}(q)$	q	1	q^3
27	$E_8(q)$	$A_7(q)$	$\Phi_2^4(q)\Phi_3^2(q)\Phi_4^2(q)\Phi_5(q)\Phi_6^3(q)\Phi_8(q)\Phi_9(q)$ $\Phi_{10}^2(q)\Phi_{12}^2(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)\Phi_{20}(q)$ $\Phi_{24}(q)\Phi_{30}(q)$	q	1	q^2
28	$E_8(q)$	$A_1(q)A_6(q)$	$\Phi_2^4(q)\Phi_3^2(q)\Phi_4^3(q)\Phi_5(q)\Phi_6^3(q)\Phi_8^2(q)\Phi_9(q)$ $\Phi_{10}^2(q)\Phi_{12}^2(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)\Phi_{20}(q)$ $\Phi_{24}(q)\Phi_{30}(q)$	q	1	q
29	$E_8(q)$	$A_1(q)A_2(q)A_4(q)$	$\Phi_2^4(q)\Phi_3^2(q)\Phi_4^3(q)\Phi_5(q)\Phi_6^4(q)\Phi_7(q)\Phi_8^2(q)$ $\Phi_9(q)\Phi_{10}^2(q)\Phi_{12}^2(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)$ $\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$	q	1	q
30	$E_8(q)$	$A_3(q)A_4(q)$	$\Phi_2^4(q)\Phi_3^2(q)\Phi_4^2(q)\Phi_5(q)\Phi_6^4(q)\Phi_7(q)\Phi_8^2(q)$ $\Phi_9(q)\Phi_{10}^2(q)\Phi_{12}^2(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)$ $\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$	q	1	q
31	$E_8(q)$	$A_2(q)D_5(q)$	$\Phi_2^3(q)\Phi_3^2(q)\Phi_4^2(q)\Phi_5(q)\Phi_6^3(q)\Phi_7(q)\Phi_8(q)$ $\Phi_9(q)\Phi_{10}^2(q)\Phi_{12}^2(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)$ $\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$	q	1	q
32	$E_8(q)$	$A_1(q)E_6(q)$	$\Phi_2^3(q)\Phi_3(q)\Phi_4^2(q)\Phi_5(q)\Phi_6^2(q)\Phi_7(q)\Phi_8(q)$ $\Phi_{10}^2(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)\Phi_{20}(q)$ $\Phi_{24}(q)\Phi_{30}(q)$	1	q	
33	$E_8(q)$	$E_7(q)$	$\Phi_2(q)\Phi_3(q)\Phi_4^2(q)\Phi_5(q)\Phi_6(q)\Phi_8(q)\Phi_{10}(q)$ $\Phi_{12}(q)\Phi_{15}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$	q	1	q

Note: $\Phi_n(q)$ is the n -th cyclotomic polynomial with $q = p^a$.

TABLE 6. Parameters for small q

X	$X \cap H$	v	k divides
${}^2B_2(8)$	$C_{13}:4$	560	156
${}^2B_2(32)$	$C_{41}:4$	198400	820
${}^3D_4(2)$	$7^2 : SL_2(3)$	179712	3528
${}^2F_4(8)$	$SU_3(8) : 2$	8004475184742400	99283968
	$PGU_3(8) : 2$	8004475184742400	99283968
$G_2(3)$	$2^3.L_3(2)$	3528	192
$G_2(4)$	$L_2(13)$	230400	2184
$G_2(5)$	J_2	416	1209600
	$G_2(2)$	484375	12096
	$2^3.L_3(2)$	4359375	1344
$G_2(7)$	$G_2(2)$	54925276	12096
$G_2(11)$	J_1	2145199320	175560
$F_4(2)$	${}^3D_4(2)$	15667200	422682624
	Alt_9	18249154560	362880
	Alt_{10}	1824915456	3628800
	$PSL_4(3) \cdot 2$	272957440	24261120
	J_2	2737373184	1209600
	$(Sym_6 \wr Sym_2).2$	3193602048	2073600
	Fi_{22}	123873281581429293827751936	64561751654400
	J_3	253925177425920	301397760
$E_6^-(2)$	Alt_{12}	319549996007424	1437004800
	$P\Omega_7(3) : 2$	16690645565440	55024220160
	Fi_{22}	1185415168	387370509926400

Then there exists a positive integer m such that $mk = \lambda f(q_0)|\text{Out}(X)|$. By Lemma 2.1(a), we have that $k(k-1) = \lambda(v-1)$, and so

$$k = \frac{m(v-1)}{f(q_0)|\text{Out}(X)|} + 1. \quad (4.6)$$

Since also k divides $|\text{Out}(X)||X \cap H|$, we conclude by (4.6) that

$$\frac{m(v-1)}{f(q_0)|\text{Out}(X)|} + 1 \mid |\text{Out}(X)||X \cap H|. \quad (4.7)$$

Now we consider the following two cases.

(i) Suppose that $(X, X \cap H)$ is as in line 1, 2, 4, 6, 8, 9, 12, 14 or 16 of Table 7. Since $m \geq 1$, it follows from (4.7) that

$$v-1 \leq f(q_0)|\text{Out}(X)|^2|X \cap H|. \quad (4.8)$$

For each case by computation the inequality (4.8) dose not hold. For example, if $X = F_4(q_0^3)$ and $X \cap H = F_4(q_0)$, then by Table 7, we have

$$v = \frac{q_0^{48}(q_0^{24} + q_0^{12} + 1)(q_0^{16} + q_0^8 + 1)(q_0^{18} - 1)}{(q_0^2 - 1)} \text{ and } f(q_0) = 2(q_0^2 - 1)^4(q_0^2 + 1)^2.$$

Note that $|X \cap H| = q_0^{24}(q_0^2 - 1)(q_0^6 - 1)(q_0^8 - 1)(q_0^{12} - 1)$. Therefore, by (4.8), we conclude that

$$q_0^{24} \leq \frac{v-1}{f(q_0)|X \cap H|} \leq |\text{Out}(X)|^2 = \gcd(2, p)^2 a^2 < 4q_0^6,$$

which is impossible.

(ii) Suppose finally that $(X, X \cap H)$ is as in line 3, 5, 7, 10, 11, 13 or 15 of Table 7. In these cases, we need extra work to do. Here by (4.7), we have that

$$m(v-1) + f(q_0)|\text{Out}(X)| \mid f(q_0)|\text{Out}(X)|^2|X \cap H|. \quad (4.9)$$

For each case, we assume that $h(q_0)$ is as in the third column Table ??, and set $d(q_0) = (v-1)h(q_0) - f(q_0)|X \cap H|$, where $f(q_0)$ is as in the fifth column of Table 7. Thus k divides

$$\begin{aligned} h(q_0)|\text{Out}(X)|[m(v-1) + f(q_0)|\text{Out}(X)|] - mf(q_0)|\text{Out}(X)|^2|X \cap H| \\ = m|\text{Out}(X)|^2[(v-1)h(q_0) - f(q_0)|X \cap H|] + f(q_0)h(q_0)|\text{Out}(X)|^3 \\ = md(q_0)|\text{Out}(X)|^2 + f(q_0)h(q_0)|\text{Out}(X)|^3. \end{aligned} \quad (4.10)$$

Therefore, (4.9) and (4.10) imply that $m(v-1) + f(q_0)|\text{Out}(X)|$ divides $md(q_0)|\text{Out}(X)|^2 + f(q_0)h(q_0)|\text{Out}(X)|^3$, and hence

$$v-1 < |\text{Out}(X)|^2[d(q_0) + f(q_0)h(q_0)|\text{Out}(X)|]. \quad (4.11)$$

For each case, this inequality holds for (p, a) as in Table 8. By applying Algorithm 1, we cannot find any possible parameters (v, k, λ) .

For example, if $X = F_4(q_0^2)$ and $X \cap H = F_4(q_0)$, then by Table 7, we have that

$$v = q_0^{24}(q_0^{12} + 1)(q_0^8 + 1)(q_0^6 + 1)(q_0^2 + 1) \text{ and } f(q_0) = (q_0^2 - 1)^4(q_0^4 + q_0^2 + 1)^2.$$

Let $h(q_0) = q_0^{16} - 4q_0^{14} + 7q_0^{12} - 12q_0^{10} + 22q_0^8 - 28q_0^6 + 31q_0^4 - 36q_0^2 + 29$ be as in Table 8. Then

$$\begin{aligned} d(q_0) &= (v-1)h(q_0) - f(q_0)|X \cap H| \\ &= 16q_0^{50} + 12q_0^{48} - 20q_0^{46} + 28q_0^{44} + 24q_0^{42} - 8q_0^{40} + 24q_0^{38} + 20q_0^{36} + 4q_0^{34} + 8q_0^{32} + \\ &\quad 36q_0^{30} - 8q_0^{28} - 4q_0^{26} + 28q_0^{24} - q_0^{16} + 4q_0^{14} - 7q_0^{12} + 12q_0^{10} - 22q_0^8 + 28q_0^6 - \\ &\quad 31q_0^4 + 36q_0^2 - 29. \end{aligned}$$

It is easily observed that (4.11) holds only for $q_0 = p^b$, where

$$\frac{p}{b} \leq \frac{2}{5} \mid \frac{3}{2} \mid \frac{5, 7}{1}$$

For each such $q_0 = p^b$, and considering each divisor k of $|F_4(q_0)| \cdot |\text{Out}(F_4(q_0^2))|$, but Algorithm 1 gives no parameters, which is contradiction. \square

4.3. Remaining cases. Recall by Corollary 2.5 that $H := G_\alpha$ is a large subgroup of G , where α is a point of \mathcal{D} . We then apply Theorem 3.1 which gives possibilities for H . By Propositions 4.1-4.2, we need to consider the remaining cases listed in Table 3. We note here that our method in this section is mostly the same as that of explained in Proposition 4.3.

Lemma 4.4. *The group X cannot be the Suzuki group ${}^2B_2(q)$, with $q = 2^{2n+1}$.*

Proof. The result follows from Propositions 4.1-4.2. \square

Lemma 4.5. *The group X cannot be the Ree group ${}^2G_2(q)$ with $q = 3^a$.*

TABLE 7. Parameters for Proposition 4.3

Line	X	$X \cap H$	v	$f(q_0)$
1	${}^2B_2(q_0^3)$	${}^2B_2(q_0)$	$q_0^4 \Phi_{12}(q_0) \Phi_3(q_0)$	$2\Phi_4(q_0)$
2	${}^2G_2(q_0^3)$	${}^2G_2(q_0)$	$q_0^6 \Phi_{18} \Phi_3(q_0)$	$4\Phi_6(q_0)$
3	${}^3D_4(q_0^2)$	${}^3D_4(q_0)$	$q_0^{12} \Phi_{24}(q_0) \Phi_4^2(q_0) \Phi_{12}(q_0)$	$81(16q_0 - 25)^2$
4	${}^2F_4(q_0^3)$	${}^2F_4(q_0)$	$q_0^{24} \Phi_3^2(q_0) \Phi_6(q_0) \Phi_{12}(q_0) \Phi_{18}(q_0) \Phi_{36}(q_0)$	$4096\Phi_4(q_0)^2$
5	$G_2(q_0^2)$	$G_2(q_0)$	$q_0^6 \Phi_4^2(q_0) \Phi_{12}(q_0)$	$81(4q_0^2 - 1)$
6	$G_2(q_0^3)$	$G_2(q_0)$	$q_0^{12} \Phi_3(q_0) \Phi_6(q_0) \Phi_9(q_0) \Phi_{18}(q_0)$	$\Phi_1^2(q_0) \Phi_2^2(q_0)$
7	$F_4(q_0^2)$	$F_4(q_0)$	$q_0^{24} \Phi_4^2(q_0) \Phi_8(q_0) \Phi_{12}(q_0)$ $\Phi_{16}(q_0) \Phi_{24}(q_0)$	$\Phi_1^4(q_0) \Phi_2^4(q_0) \Phi_3^2(q_0) \Phi_6^2(q_0)$
8	$F_4(q_0^3)$	$F_4(q_0)$	$q_0^{48} \Phi_3^2(q_0) \Phi_6^2(q_0) \Phi_{12}(q_0) \Phi_9^2(q_0)$ $\Phi_{18}^2(q_0) \Phi_{24}(q_0) \Phi_{36}(q_0)$	$2\Phi_1(q_0)^4 \Phi_2^4(q_0) \Phi_4^2(q_0)$
9	$E_6^-(q_0^3)$	$E_6^-(q_0)$	$q_0^{72} \Phi_3^2(q_0) \Phi_6^3(q_0) \Phi_9^2(q_0) \Phi_{12}(q_0)$ $\Phi_{18}^2(q_0) \Phi_{24}(q_0) \Phi_{30}(q_0) \Phi_{36}(q_0) \Phi_{54}(q_0)$	$\Phi_1^4(q_0) \Phi_2^6(q_0) \Phi_{10}(q_0) \Phi_4^2(q_0) \Phi_8(q_0)$
10	$E_6(q_0^2)$	$E_6(q_0)$	$q_0^{36} \Phi_2^2(q_0) \Phi_4^2(q_0) \Phi_6(q_0) \Phi_8(q_0)$ $\Phi_{10}(q_0) \Phi_{12}(q_0) \Phi_{16}(q_0) \Phi_{18}(q_0) \Phi_{24}(q_0)$	$\Phi_1^6(q_0) \Phi_5(q_0) \Phi_3^3(q_0) \Phi_9(q_0)$
11	$E_6(q_0^2)$	$E_6^-(q_0)$	$q_0^{36} \Phi_1^2(q_0) \Phi_3(q_0) \Phi_4^2(q_0) \Phi_5(q_0)$ $\Phi_8(q_0) \Phi_9(q_0) \Phi_{12}(q_0) \Phi_{16}(q_0) \Phi_{24}(q_0)$	$\Phi_2(q_0)^6 \Phi_3^3(q_0) \Phi_{10}(q_0) \Phi_{18}(q_0)$
12	$E_6(q_0^3)$	$E_6(q_0)$	$q_0^{72} \Phi_3^3(q_0) \Phi_6^2(q_0) \Phi_9^2(q_0) \Phi_{18}^2(q_0)$ $\Phi_{12}(q_0) \Phi_{15}(q_0) \Phi_{24}(q_0) \Phi_{27}(q_0) \Phi_{36}(q_0)$	$\Phi_1^6(q_0) \Phi_2^4(q_0) \Phi_5(q_0) \Phi_4^2(q_0) \Phi_8(q_0)$
13	$E_7(q_0^2)$	$E_7(q_0)$	$q_0^{63} \Phi_4^5(q_0) \Phi_8(q_0) \Phi_{12}^2(q_0) \Phi_{16}(q_0)$ $\Phi_{20}(q_0) \Phi_{24}(q_0) \Phi_{28}(q_0) \Phi_{36}(q_0)$	$3\Phi_1^7(q_0) \Phi_2^7(q_0) \Phi_3^3(q_0) \Phi_6^3(q_0) \Phi_5(q_0)$ $\Phi_{10}(q_0) \Phi_9(q_0) \Phi_{18}(q_0)$
14	$E_7(q_0^3)$	$E_7(q_0)$	$q_0^{126} \Phi_3^4(q_0) \Phi_6^4(q_0) \Phi_9^2(q_0)$ $\Phi_{12}(q_0) \Phi_{15}(q_0) \Phi_{18}^2(q_0) \Phi_{27}(q_0) \Phi_{21}(q_0)$ $\Phi_{24}(q_0) \Phi_{30}(q_0) \Phi_{36}(q_0) \Phi_{42}(q_0) \Phi_{54}(q_0)$	$8\Phi_1^7(q_0) \Phi_2^7(q_0) \Phi_4^2(q_0) \Phi_5(q_0) \Phi_{10}(q_0)$
15	$E_8(q_0^2)$	$E_8(q_0)$	$q_0^{120} \Phi_4^4(q_0) \Phi_8^2(q_0) \Phi_{12}^2(q_0) \Phi_{16}^2(q_0)$ $\Phi_{20}(q_0) \Phi_{24}(q_0) \Phi_{28}(q_0) \Phi_{36}(q_0) \Phi_{40}(q_0)$ $\Phi_{48}(q_0) \Phi_{60}(q_0)$	$\Phi_1^8(q_0) \Phi_2^8(q_0) \Phi_3^4(q_0) \Phi_5^2(q_0)$ $\Phi_6^4(q_0) \Phi_7(q_0) \Phi_9(q_0) \Phi_{14}(q_0) \Phi_{10}^2(q_0)$ $\Phi_{15}(q_0) \Phi_{18}(q_0) \Phi_{30}(q_0)$
16	$E_8(q_0^3)$	$E_8(q_0)$	$q_0^{240} \Phi_3^4(q_0) \Phi_6^4(q_0) \Phi_9^3(q_0) \Phi_{12}^2(q_0)$ $\Phi_{15}(q_0) \Phi_{18}^3(q_0) \Phi_{21}(q_0) \Phi_{24}(q_0) \Phi_{27}(q_0)$ $\Phi_{30}(q_0) \Phi_{36}^2(q_0) \Phi_{42}(q_0) \Phi_{45}(q_0) \Phi_{54}(q_0)$ $\Phi_{60}(q_0) \Phi_{72}(q_0) \Phi_{90}(q_0)$	$\Phi_1^8(q_0) \Phi_2^8(q_0) \Phi_4^4(q_0) \Phi_5^2(q_0) \Phi_7(q_0)$ $\Phi_8^2(q_0) \Phi_{10}^2(q_0) \Phi_{14}(q_0) \Phi_{20}(q_0)$

Note: $\Phi_n(q)$ is the n -th cyclotomic polynomial with $q = p^a$.

Proof. By Propositions 4.1-4.3, the subgroup $X \cap H$ is $2 \times PSL_2(q)$ for $q \geq 27$. Then (4.1) implies that $v = q^2(q^2 - q + 1)$. Note that $|\text{Out}(X)| = a$. Then by Lemma 2.1(b), k must divide $aq(q^2 - 1)$. Moreover, Lemma 2.4(a) implies that k divides $\lambda(v - 1)$. By Tits' lemma 2.2, $v - 1$ is coprime to q , and since $\gcd(v - 1, q^2 - 1) = q - 1$, it follows that k divides $\lambda a(q - 1)$. Therefore there exists a positive integer m such that $mk = \lambda a(q - 1)$. Since now $k(k - 1) = \lambda(v - 1)$, we must have

$$\frac{\lambda a(q - 1)}{m}(k - 1) = \lambda(q^4 - q^3 + q^2 - 1).$$

Thus,

$$k = \frac{m(q^3 + q + 1)}{a} + 1 \quad (4.12)$$

Since $k \mid aq(q^2 - 1)$, it follows from (4.12) that

$$m(q^3 + q + 1) + a \mid a^2 q(q^2 - 1). \quad (4.13)$$

Note that $a^2[m(q^3 + q + 1) + a] - ma^2 q(q^2 - 1) = ma^2(2q + 1) + a^3$. Then by (4.13), we conclude that $m(q^3 + q + 1) + a$ divides $ma^2(2q + 1) + a^3$, and so $m(q^3 + q + 1) + a < ma^2(2q + 1) + a^3 < 2ma^3(q + 1)$. Hence $q^3 + q + 1 < 2a^3(q + 1)$, which is impossible. \square

Lemma 4.6. *The group X cannot be ${}^3D_4(q)$.*

TABLE 8. Some parameters for Proposition 4.3

X	$X \cap H$	$h(q_0)$	Conditions on $q_0 = p^b$
${}^3D_4(q_0^2)$	${}^3D_4(q_0)$	$20736q_0^4 - 106272q_0^2 + 263169$	-
$G_2(q_0^2)$	$G_2(q_0)$	$324q_0^2 - 729$	$p = 2$ and $b \leq 7$ $p = 3$ and $b \leq 5$ $p = 5, 7$ and $b \leq 2$ $p = 11, \dots, 23$ and $b = 1$
$F_4(q_0^2)$	$F_4(q_0)$	$q_0^{16} - 4q_0^{14} + 7q_0^{12} - 12q_0^{10} + 22q_0^8 - 28q_0^6 + 31q_0^4 - 36q_0^2 + 29$	$p = 2$ and $b \leq 5$ $p = 3$ and $b \leq 2$ $p = 5, 7$ and $b = 1$
$E_6(q_0^2)$	$E_6^\epsilon(q_0)$	$q_0^{22} - \epsilon 2q_0^{21} - q_0^{20} + \epsilon 2q_0^{19} + 4q_0^{18} - \epsilon 5q_0^{17} - 3q_0^{16} + \epsilon 9q_0^{15} + q_0^{14} - \epsilon 15q_0^{13} + 2q_0^{12} + \epsilon 22q_0^{11} - 14q_0^{10} - \epsilon 25q_0^9 + 27q_0^8 + \epsilon 27q_0^7 - 47q_0^6 - \epsilon 13q_0^5 + 68q_0^4 - \epsilon 8q_0^3 - 83q_0^2 + \epsilon 44q_0 + 81$	$p = 2$ and $b \leq 20$ $p = 3$ and $b \leq 9$ $p = 5, 7$ and $b \leq 6$ $p = 11, 13$ and $b \leq 4$ $p = 17$ and $b \leq 2$ $p = 19$ and $b \leq 3$ $p = 23, \dots, 113$ and $b \leq 2$ $p = 127, \dots, 3307$ and $b = 1$
$E_7(q_0^2)$	$E_7(q_0)$	$3q_0^{46} - 15q_0^{44} + 33q_0^{42} - 57q_0^{40} + 102q_0^{38} - 159q_0^{36} + 222q_0^{34} - 306q_0^{32} + 396q_0^{30} - 516q_0^{28} + 681q_0^{26} - 858q_0^{24} + 1092q_0^{22} - 1383q_0^{20} + 1704q_0^{18} - 2076q_0^{16} + 2478q_0^{14} - 2934q_0^{12} + 3459q_0^{10} - 4056q_0^8 + 4719q_0^6 - 5451q_0^4 + 6285q_0^2 - 7149$	$p = 2$ and $b \leq 9$ $p = 3$ and $b \leq 6$ $p = 5$ and $b \leq 4$ $p = 7$ and $b \leq 3$ $p = 11, 13, 17$ and $b \leq 2$ $p = 19, \dots, 179$ and $b = 1$
$E_8(q_0^2)$	$E_8(q_0)$	$q_0^{88} - 4q_0^{86} + 7q_0^{84} - 10q_0^{82} + 14q_0^{80} - 15q_0^{78} + 13q_0^{76} - 10q_0^{74} + 9q_0^{72} - 14q_0^{70} + 22q_0^{68} - 32q_0^{66} + 39q_0^{64} - 36q_0^{62} + 20q_0^{60} + q_0^{58} - 10q_0^{56} + 4q_0^{54} + 22q_0^{52} - 58q_0^{50} + 79q_0^{48} - 66q_0^{46} + 16q_0^{44} + 42q_0^{42} - 73q_0^{40} + 46q_0^{38} + 34q_0^{36} - 132q_0^{34} + 186q_0^{32} - 139q_0^{30} + 8q_0^{28} + 140q_0^{26} - 209q_0^{24} + 136q_0^{22} + 62q_0^{20} - 278q_0^{18} + 365q_0^{16} - 234q_0^{14} - 71q_0^{12} + 377q_0^{10} - 486q_0^8 + 290q_0^6 + 135q_0^4 - 548q_0^2 + 661$	$p = 2$ and $b \leq 7$ $p = 3$ and $b \leq 3$ $p = 5$ and $b \leq 2$ $p = 7, 11, 13, 17$ and $b = 1$

Proof. By Theorem 3.1 and Propositions 4.1-4.2, $X \cap H$ is one of the following groups:

- (i) $G_2(q)$;
- (ii) $PSL_2(q^3) \times PSL_2(q)$ with q even;
- (iii) $(SL_2(q^3) \circ SL_2(q)) \cdot 2$ with q odd;
- (iv) $(SL_3^\epsilon(q) \circ (q^2 + \epsilon 1q + 1)) \cdot d_\epsilon \cdot 2$ with $d_\epsilon = \gcd(3, q^2 + \epsilon 1q + 1)$ for $\epsilon = \pm$.

(i) Suppose $X \cap H = G_2(q)$. Then by (4.1), we have that $v = q^6(q^8 + q^4 + 1)$. Here $|\text{Out}(X)| = 3a$, and so Lemma 2.1(b) implies that k divides $3aq^6(q^6 - 1)(q^2 - 1)$. Moreover, k divides $\lambda(v - 1)$ by Lemma 2.4(a), and so by Tit's lemma 2.2, k must divide $\gcd((q^6 - 1)(q^2 - 1), v - 1)$ which is a divisor of 4. Thus $k \mid 12\lambda a$, and hence there exists a positive integer m such that $mk = 12\lambda a$. As $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(q^{14} + q^{10} + q^6 - 1)}{12a} + 1, \quad (4.14)$$

and since $k \mid 3aq^6(q^6 - 1)(q^2 - 1)$ by Lemma 2.1(b), we conclude that

$$m(q^{14} + q^{10} + q^6 - 1) + 12a \mid 36a^2q^6(q^6 - 1)(q^2 - 1). \quad (4.15)$$

Note that

$$\begin{aligned} & 36a^2[m(q^{14} + q^{10} + q^6 - 1) + 12a] - 36ma^2q^6(q^6 - 1)(q^2 - 1) \\ & = 36ma^2(q^{12} + q^{10} + q^8 - 1) + 432a^3. \end{aligned}$$

Then by (4.15), we must have that $m(q^{14} + q^{10} + q^6 - 1) + 12a$ divides $36ma^2(q^{12} + q^{10} + q^8 - 1) + 432a^3$, and so

$$m(q^{14} + q^{10} + q^6 - 1) + 12a < 36ma^3(q^{12} + q^{10} + q^8 + 11).$$

Therefore, $m(q^{14} + q^{10} + q^6 - 1) < 36ma^3(q^{12} + q^{10} + q^8 + 11)$, and hence

$$q^2 - 1 < \frac{q^{14} + q^{10} + q^6 - 1}{q^{12} + q^{10} + q^8 + 11} < 36a^3.$$

This inequality holds when

$$\frac{p}{a} \leq \begin{array}{c|ccc} & 2 & 3 & 5 \\ \hline & 6 & 3 & 1 \end{array}$$

For these values of $q = p^a$, we observe that $m \leq 1295$ since otherwise, $36a^2q^6(q^6 - 1)(q^2 - 1) < 1296(q^{14} + q^{10} + q^6 - 1) < m(q^{14} + q^{10} + q^6 - 1)$ which contradicts (4.15). However, such possible values of q and m do not give any parameter k from (4.14), which is a contradiction.

(ii) Suppose $X \cap H = PSL_2(q^3) \times PSL_2(q)$ with q even. Then by (4.1), we have that $v = q^8(q^8 + q^4 + 1)$. As $|\text{Out}(X)| = 3a$, it follows from Lemma 2.1(b) that k divides $3aq^4(q^6 - 1)(q^2 - 1)$, and so by Lemma 2.4(a) and Tit's lemma 2.2, k divides $3\lambda \gcd((q^6 - 1)(q^2 - 1), v - 1)a = 3\lambda a$, and hence there exists a positive integer m such that $mk = 3\lambda a$. Since $k(k - 1) = \lambda(v - 1)$, we have that $k = (1/3a)m(q^{16} + q^{12} + q^8 - 1) + 1$, and so by Lemma 2.1(b), we must have $m(q^{16} + q^{12} + q^8 - 1) + 3a \mid 9a^2q^4(q^6 - 1)(q^2 - 1)$. Therefore, $m(q^{16} + q^{12} + q^8 - 1) + 3a \leq 9a^2q^4(q^6 - 1)(q^2 - 1)$, and hence $q^4 < (q^{16} + q^{12} + q^8 - 1)(q^4(q^6 - 1)(q^2 - 1)) < 9a^2$, which is impossible for $q = 2^a$.

(iii) Suppose $X \cap H = (SL_2(q^3) \circ SL_2(q)) \cdot 2$ with q odd. Then by (4.1), we have that $v = q^8(q^8 + q^4 + 1)$. By Lemma 2.1(b), k divides $3aq^4(q^6 - 1)(q^2 - 1)$ as $|\text{Out}(X)| = 3a$. Now by Lemma 2.4(a), k divides $\lambda(v - 1)$, and so by applying Tit's lemma 2.2, k divides $3a \gcd((q^6 - 1)(q^2 - 1), v - 1)$ which divides $12a$. Thus there exists a positive integer m such that $mk = 12\lambda a$. Since $k(k - 1) = \lambda(v - 1)$, it follows that $k = (1/12a)m(q^{16} + q^{12} + q^8 - 1) + 1$, and since $k \mid 3aq^4(q^6 - 1)(q^2 - 1)$, we have that

$$m(q^{16} + q^{12} + q^8 - 1) + 12a \mid 36a^2q^4(q^6 - 1)(q^2 - 1).$$

Therefore, $m(q^{16} + q^{12} + q^8 - 1) + 12a \leq 36a^2q^4(q^6 - 1)(q^2 - 1)$, and hence

$$q^4 < \frac{q^{16} + q^{12} + q^8 - 1}{q^4(q^6 - 1)(q^2 - 1)} < 36a^2,$$

which is a contradiction for odd $q = p^a$.

(iv) Suppose $X \cap H = (SL_3^{\epsilon}(q) \circ (q^2 + \epsilon 1q + 1)) \cdot d_{\epsilon} \cdot 2$ where $d_{\epsilon} = \gcd(3, q^2 + \epsilon 1q + 1)$ with $\epsilon = \pm$. Then by (4.1), we have that

$$v = \frac{q^9(q^8 + q^4 + 1)(q^3 + \epsilon 1)}{2 \cdot (q^2 + \epsilon 1q + 1) \cdot d_{\epsilon}}.$$

Since $|\text{Out}(X)| = 3a$, it follows from Lemma 2.1(b) that k divides $6a \cdot d_{\epsilon} \cdot q^3(q^3 - \epsilon 1)(q^2 - 1)(q^2 + \epsilon 1q + 1)$. Since also $\gcd((q^3 - \epsilon 1)(q^2 - 1), 2d_{\epsilon}(v - 1)) = \gcd((q^3 - \epsilon 1)(q + 1), 2d_{\epsilon}(v - 1))$ divides n where n is as in the third column of Table 4.3, we conclude by Lemma 2.4(a) and Tits' lemma 2.2 that $k \mid 6an \cdot d_{\epsilon} \cdot \lambda(q^2 + \epsilon 1q + 1)$.

Thus there exists a positive integer m such that $mk = 6an \cdot d_\epsilon \cdot \lambda(q^2 + \epsilon 1q + 1)$. As $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{6an \cdot d_\epsilon \cdot (q^2 + \epsilon 1q + 1)} + 1. \quad (4.16)$$

By Lemma 2.1(b), k divides $6a \cdot d_\epsilon \cdot q^3(q^3 - \epsilon 1)(q^2 - 1)(q^2 + \epsilon 1q + 1)$. Then, by (4.16), we must have

$$m(v-1) + 6an \cdot d_\epsilon \cdot t(q) \leq 36a^2n \cdot d_\epsilon^2 \cdot q^3(q^3 - \epsilon 1)(q^2 - 1)t(q)^2,$$

where $t(q) := q^2 + \epsilon 1q + 1$ with $\epsilon = \pm$. Let now $l(q)$ and $u(a)$ be as in the third column of Table 4.3. Then

$$l(q) \leq \frac{d_\epsilon \cdot (v-1)}{nq^3(q^3 - \epsilon 1)(q^2 - 1)(q^2 + \epsilon 1q + 1)^2} \leq u(a). \quad (4.17)$$

TABLE 9. Some parameters for Lemma 4.6(iii)

ϵ	d_ϵ	n	$l(q)$	$u(q)$	Conditions on p and a
\pm	1	$3(q - \epsilon 1)^2$	q^4	$216a^2$	$p = 2$ and $a \in \{1, 2\}$ $p = 3$ and $a = 1$
\pm	3	8	q^6	$15552a^2$	$p = 2$ and $a \in \{1, 2\}$ $p = 3$ and $a = 1$

In each case as in Table 4.3, the inequality (4.17) holds when p and a are listed in the last column of Table 4.3. Note that $(\epsilon, q, d_\epsilon)$ can not be $(+, 4, 3)$ and $(-, 2, 3)$ as in these cases v is not integer. For the remaining (p, a) , considering the condition d_ϵ as in the second column, we obtain $(\epsilon, q, d_\epsilon)$ as in Table 10. Recall that k is a divisor of $6a \cdot d_\epsilon \cdot q^3(q^3 - \epsilon 1)(q^2 - 1)(q^2 + \epsilon 1q + 1)$ by Lemma 2.1(b). Then, for each $q = p^a$ with p and a as in the last column of Table 4.3, the possible values of k and v are listed in Table 10. This is a contradiction as for each k and v as in Table 10, the fraction $k(k-1)/(v-1)$ is not integer. This also could be done by Algorithm 1. \square

TABLE 10. Possible values for k and v for Lemma 4.6(iii).

ϵ	q	d_ϵ	v	k divides
+	2	1	89856	7056
+	3	1	140812182	438048
-	4	1	41791389696	9734400
-	3	1	242829171	254016

Lemma 4.7. *If $X = G_2(q)$, then $X \cap H$ is isomorphic to $SL_3^\epsilon(q) : 2$ with $\epsilon = \pm$ and (v, k, λ) is as in Table 1.*

Proof. Suppose $X = G_2(q)$. Then Theorem 3.1 and Propositions 4.1-4.2, $X \cap H$ is one of the following groups:

- (i) $d \cdot PSL_2(q)^2 : d$ with $d = \gcd(2, q-1)$;
- (ii) ${}^2G_2(q)$ with $q = 3^{2n+1} \geq 27$;
- (iii) $SL_3^\epsilon(q) : 2$ with $\epsilon = \pm$.

(i) Suppose $X \cap H = d \cdot PSL_2(q)^2 : d$ with $d = \gcd(2, q-1)$. Then by (4.1), we have that $v = q^4(q^4 + q^2 + 1)$. Since $|\text{Out}(X)|$ divides $2a$, by Lemmas 2.4(d) and 2.1(b), $k \mid 2aq^2(q^2 - 1)^2$. Moreover Lemma 2.4(a) implies that, k divides $\lambda(v-1)$. By Tit's lemma 2.2, $v-1$ is coprime to q , and $\gcd(q^2 - 1, v-1)$ divides 4. Thus $k \mid 32\lambda a$. Let m be a positive integer such that $mk = 32\lambda a$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{32a} + 1. \quad (4.18)$$

Note by Lemma 2.4(c) that k divides $2aq^2(q^2 - 1)^2$. Then by (4.18), we must have:

$$m(v-1) + 32a \mid 64a^2q^2(q^2 - 1)^2. \quad (4.19)$$

Therefore $m(v-1) + 32a \leq 64a^2q^2(q^2 - 1)^2$, and so we have that

$$q^2 < \frac{q^2(q^4 + q^2 + 1)}{(q^2 - 1)^2} \leq 64a^2.$$

This inequality holds when p is 2, 3, 5 or 7 and a is at most 5, 2, 1 or 1, respectively, and so $q = p^a \in \{2, 3, 4, 5, 7, 8, 9, 16, 32\}$. For these p and a , if $m \geq 14$, then we would have $m(v-1) \geq 14q^4(q^4 + q^2 + 1) - 1 > 64a^2q^2(q^2 - 1)^2$ which contradicts (4.19). Therefore, $m \leq 13$, and hence, for each m and q the parameter k as in (4.18) is not a positive integer, which is a contradiction.

(ii) Suppose $X \cap H = {}^2G_2(q)$ with $q = 3^{2n+1} \geq 27$. Then by (4.1) we have that $v = q^3(q+1)(q^3-1)$. We observe that $\gcd(q^2-1, v-1) = 1$, and since k divides both $2aq^3(q-1)(q^3+1)$ and $\lambda(v-1)$, by Tit's lemma 2.2, we conclude that $k \mid 2\lambda af(q)$, where $f(q) = q^2 - q + 1$. Let m be a positive integer such that $mk = 2\lambda af(q)$. It follows from $k(k-1) = \lambda(v-1)$ that

$$k = \frac{m(v-1)}{2af(q)} + 1 \quad (4.20)$$

Recall that $k \mid 2aq^3(q-1)(q^3+1)$. Then, by (4.20), we must have

$$m(v-1) + 2af(q) \mid 4a^2f(q)q^3(q-1)(q^3+1). \quad (4.21)$$

Let $h(q) = q^2 - 3q + 5$ and $d(q) = 4q^6 + 4q^5 - 4q^4 - 4q^3 - q^2 + 3q - 5$. Then

$$4a^2h(q)[m(v-1) + 2af(q)] - 4ma^2f(q) \cdot q^3(q^3+1)(q-1) = 4ma^2d(q) + 8a^3f(q)h(q),$$

where $f(q) = q^2 - q + 1$. Therefore (4.21) implies that

$$\begin{aligned} m(v-1) + 2af(q) &\leq 4ma^2d(q) + 8a^3f(q)h(q) \\ &< 4ma^2(d(q) + 2af(q)h(q)), \end{aligned}$$

and so

$$\frac{q}{4} < \frac{q^3(q^3-1)(q+1)}{d(q) + 2f(q)h(q)} \leq 4a^3.$$

This inequality holds when $q = p^a$ is as in Table 11. Since $\lambda < k$, from the fact that $mk = 2af(q) = 2a(q^2 - q + 1)$, we observe that $m \leq 2a(q^2 - q + 1)$, and so, for each q and a as in Table 11, we can find an upper bound for m as listed in the same table. Thus for each such q , a and m , we obtain k by (4.20), and considering the fact that $k(k-1) = \lambda(v-1)$, we conclude that (q, a, m) is (3, 1, 6) or

TABLE 11. Possible values for q , a and m in Lemma 4.7(ii).

q	a	$m \leq$	q	a	$m \leq$	q	a	$m \leq$
2	1	6	32	5	9930	729	6	6368556
3	1	14	49	2	9412	1024	10	20951060
4	2	52	64	6	48396	2048	11	92229654
5	1	42	81	4	51848	2187	7	66930962
7	1	86	121	2	58084	4096	12	402554904
8	3	342	125	3	93006	6561	8	688642576
9	2	292	128	7	227598	8192	13	1744617498
11	1	222	243	5	588070	16384	14	7515734044
13	1	314	256	8	1044496	32768	15	32211271710
16	4	1928	343	3	703842	65536	16	137436856352
25	2	2404	512	9	4709394			
27	3	4218	625	4	3120008			

$(2187, 7, 28684698)$, and hence (v, k, λ) is one of the following parameter, however k has to divide $2aq^3(q-1)(q^3+1)$, that is to say, 3024 or 3348658743700932890656848, respectively, which is impossible.

v	k	λ
2808	1204	516
239408748196861789141128	102603749227226481060484	43973035383097063311636

(iii) Suppose $X \cap H = SL_3^\epsilon(q) : 2$ with $\epsilon = \pm$. Then by (4.1), we have that

$$v = \frac{q^3(q^3 + \epsilon 1)}{2}.$$

Thus $v - 1 = (q^3 - \epsilon 1)(q^3 + \epsilon 2)/2$. By [39, Lemma 15] and [38, Sections 4-5] and Lemma 2.4(e), we conclude that k divides $\lambda d_\epsilon(q^3 - \epsilon 1)$, where $d_\epsilon = 1$ if q is odd and $d_\epsilon = \gcd(q - 2, q^2 + \epsilon 1)$ if q is even ($d_- = 3$ or $d_+ = 5$). Therefore, there exists a positive integer m such that $mk = \lambda d_\epsilon(q^3 - \epsilon 1)$. Since $\lambda < k$ we have

$$m < d_\epsilon(q^3 - \epsilon 1). \quad (4.22)$$

As $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(q^3 + \epsilon 2) + 2d_\epsilon}{2d_\epsilon}, \quad (4.23)$$

$$\lambda = \frac{m^2(q^3 + \epsilon 2) + 2md_\epsilon}{2d_\epsilon^2(q^3 - \epsilon 1)}, \quad (4.24)$$

where $d_\epsilon = 1$ if q is odd and if q is even with $d_- = 3$ or $d_+ = 5$. By (4.24), we conclude that $q^3 - \epsilon 1$ divides $m(3m + \epsilon 2d_\epsilon)$, and so $q^3 - \epsilon 1$ divides $2d_\epsilon m$ or $3m + \epsilon 2d_\epsilon$.

Suppose first $q^3 - \epsilon 1$ divides $2d_\epsilon m$. Then $m = c(q^3 - \epsilon 1)/2d_\epsilon$ for some positive integer c . We observe by (4.22) that $c \in \{1, 2, d_\epsilon, d_\epsilon^2, 2d_\epsilon, 2d_\epsilon^2\}$, and so (4.23) and (4.24) imply that

$$k = \frac{c(q^3 - \epsilon 1)(q^3 + \epsilon 2)}{4d_\epsilon^2} + 1 \text{ and } \lambda = \frac{c^2(q^3 - \epsilon 1)(q^3 + \epsilon 2) + 4cd_\epsilon^2}{8d_\epsilon^4}.$$

If q is odd, then $d_e = 1$, and since $c(q^3 - \epsilon 1)(q^3 + \epsilon 2)$ must be a multiple of 4, we conclude that $c = 1, 2$. Since also $\lambda < k$, we must have $c = 1$. Therefore,

$$k = \frac{(q^3 - \epsilon 1)(q^3 + \epsilon 2) + 4}{4} \text{ and } \lambda = \frac{(q^3 - \epsilon 1)(q^3 + \epsilon 2) + 4}{8}.$$

If q is even, then since 4 divides $c(q^3 - \epsilon 1)(q^3 + \epsilon 2)$ and $(q^3 + \epsilon 2)_2 = 2$, we conclude that c is even, and this implies that $c \in \{2, 2d_e, 2d_e^2\}$. The case where $c = 2d_e^2$ can be ruled out as $\lambda < k$. Therefore, $c \in \{2, 2d_e\}$ as desired.

Suppose now $q^3 - \epsilon 1$ divides $3m + \epsilon 2d_e$. Then there exists some positive integer c such that

$$m = \frac{c(q^3 - \epsilon 1) - \epsilon 2d_e}{3}, \quad (4.25)$$

where $d_e = 1$ if q is odd, and d_e is 3 or 5 if q is even respectively for $\epsilon = -$ or $+$.

If $c \geq 4$, then $m \geq (4(q^3 - \epsilon 1) - 10)/3 > q^3 - \epsilon 1$, which contradicts (4.22). Thus $c \in \{1, 2, 3\}$ and one of the following holds:

- (1) $m = [cq^3 - \epsilon(c + 2)]/3$ if q odd;
- (2) $m = (cq^3 - c - 10)/3$ if q even and $\epsilon = +$;
- (3) $m = (cq^3 + c + 6)/3$ if q even and $\epsilon = -$.

Let first (1) hold. Note that $c = 3$ never holds as $m = (3q^3 - 5)/3$ is not integer. If $c = 1$, then $m = (q^3 - \epsilon 3)/3$, and so we must have $q = 3^a$. By (4.23), we obtain

$$k = \frac{q^3(q^3 - \epsilon 1)}{6} \text{ and } \lambda = \frac{q^3(q^3 - \epsilon 3)}{18}. \quad (4.26)$$

If $c = 2$, then $m = (2q^3 - \epsilon 4)/3$ in which case $q \equiv -\epsilon 1 \pmod{3}$. So by (4.23), we must have

$$k = \frac{q^6 - 1}{3} \text{ and } \lambda = \frac{2(q^3 + \epsilon 1)(q^3 - \epsilon 2)}{9}. \quad (4.27)$$

Let now (2) hold. If $c = 2$ or 3, then m would be $(2q^3 - 12)/3$ or $(3q^3 - 13)/3$, respectively, which is impossible as in both cases, the nominator is not divisible by 3. Therefore, $c = 1$. Thus $m = (q^3 - 11)/3$ in which case $q = 2^a$ with a odd. By (4.23), we must have

$$k = \frac{(q^3 - 1)(q^3 - 8)}{30} \text{ and } \lambda = \frac{(q^3 - 8)(q^3 - 11)}{450}. \quad (4.28)$$

Let finally (3) hold. If $c = 1$, then $m = (q^3 + 7)/3$ where $q = 2^a$ with a odd. By (4.23), we obtain

$$k = \frac{(q^3 + 1)(q^3 + 4)}{18} \text{ and } \lambda = \frac{(q^3 + 4)(q^3 + 7)}{162}. \quad (4.29)$$

If $c = 2$, then $m = (2q^3 + 8)/3$ where $q = 2^a$ with a odd. So by (4.23), we must have

$$k = \frac{(q^3 + 1)^2}{9} \text{ and } \lambda = \frac{(q^3 + 1)(q^3 + 4)}{81}. \quad (4.30)$$

If $c = 3$, then $m = q^3 + 3$ where $q = 2^a$, and so (4.23) implies that

$$k = \frac{q^3(q^3 + 1)}{6} \text{ and } \lambda = \frac{q^3(q^3 + 3)}{18}. \quad (4.31)$$

□

Lemma 4.8. *The group X cannot be ${}^2F_4(q)$.*

Proof. Suppose $X = {}^2F_4(q)$. Then by Theorem 3.1, $X \cap H$ is one of the following:

- (i) ${}^2B_2(q) \wr 2$ with $q = 2^{2n+1} \geq 8$;
 - (ii) $O_5(q):2$, with $q = 2^{2n+1} \geq 8$.
- (i) Suppose $X \cap H = {}^2B_2(q) \wr 2$ with $q = 2^{2n+1} \geq 8$. Then by (4.1), we have that $v = \frac{q^8(q^6+1)(q^3+1)(q+1)}{2(q^2+1)}$. We observe that $\gcd((q^2+1)^2(q-1)^2, v-1)$ divides 4, and since k divides both $2aq^4(q^2+1)^2(q-1)^2$ and $\lambda(v-1)$, by Tit's lemma 2.2, we conclude that $k \mid 8\lambda a$. Let m be a positive integer such that $mk = 8\lambda a$. It follows from $k(k-1) = \lambda(v-1)$ that

$$k = \frac{m(v-1)}{8a} + 1. \quad (4.32)$$

Recall that $k \mid 2aq^4(q^2+1)^2(q-1)^2$. Then, by (4.32), we must have

$$m(v-1) + 8a \mid 16a^2q^4(q^2+1)^2(q-1)^2. \quad (4.33)$$

Since $\frac{v-1}{q^4(q^2+1)^2(q-1)^2} \geq q^6$, (4.33) implies that $q^6 \leq 32a^2$ where $q = p^a$, which is a contradiction.

- (ii) Now suppose $X \cap H = O_5(q):2$ with $q = 2^{2n+1} \geq 8$. Then by (4.1), we have that $v = 2q^8(q^6+1)(q^2-q+1)$. Since $\gcd(q^2+1, v-1) = 1$ and k divides both $\frac{aq^4(q^2-1)^2(q^2+1)}{2}$ and $\lambda(v-1)$, by Tit's lemma 2.2, we conclude that $k \mid \frac{\lambda a(q^2-1)^2}{2}$. Let m be a positive integer such that $mk = \frac{\lambda a(q^2-1)^2}{2}$. It follows from $k(k-1) = \lambda(v-1)$ that $k = 2m(v-1)/a(q^2-1)^2 + 1$, and since k divides $2aq^4(q^2+1)^2(q-1)^2$, we must have

$$2m(v-1) + a(q^2-1)^2 \mid a^2q^4(q^2-1)^4(q^2+1). \quad (4.34)$$

Since now $\frac{v-1}{q^4(q^2-1)^4(q^2+1)} \geq q^2$, we conclude by (4.34) that $q^2 \leq a^2$, which is a contradiction. \square

Lemma 4.9. *If X is $F_4(q)$, then H is as in Table 2.*

Proof. Suppose $X = F_4(q)$. Theorem 3.1 and Propositions 4.1-4.2, $X \cap H$ is one of the following groups:

- (i) $2 \cdot (PSL_2(q) \times PSp_6(q)) \cdot 2$ with q odd;
 - (ii) $2 \cdot \Omega_9(q)$;
 - (iii) $PSp_8(q)$ with $p = 2$;
 - (iv) $d^2 \cdot D_4(q) \cdot Sym_3$ with $d = \gcd(2, q-1)$;
 - (v) ${}^3D_4(q) \cdot 3$;
 - (vi) $Sp_4(q^2) \cdot 2$ with $p = 2$;
 - (vii) $Sp_4(q)^2 \cdot 2$ with $p = 2$;
 - (viii) ${}^2F_4(q)$ with $q = 2^{2n+1} \geq 2$;
 - (ix) $\text{Soc}(H) = PSL_2(q) \times G_2(q)$ with $q > 3$ odd and $G \neq X$.
- (i) Suppose $X \cap H = 2 \cdot (PSL_2(q) \times PSp_6(q)) \cdot 2$ with q odd. Then $|X \cap H| = q^{10}(q^2-1)^2(q^4-1)(q^6-1)$, and so by (4.1) implies that

$$v = q^{14}(q^6 + q^4 + q^2 + 1)(q^8 + q^4 + 1).$$

As in this case q is odd, $|\text{Out}(X)| = a$. Hence by Lemma 2.4(c), we have that

$$k \mid aq^{10}(q^6-1)(q^4-1)(q^2-1)^2. \quad (4.35)$$

Moreover, Lemma 2.4(a) implies that k divides $\lambda(v-1)$. By Tit's lemma 2.2, $v-1$ is coprime to q , and since $\gcd((q^4+q^2+1)(q^2+1), v-1) = 1$ and $\gcd(v-1, q^2-1)$

divides 11, k must divide $11^4 a \lambda$. Therefore there exists a positive integer m such that $mk = 14641 \lambda a$. As $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{14641a} + 1. \quad (4.36)$$

Applying (4.35) and (4.36), we have that

$$m(v-1) + 14641a \mid 14641a^2 q^{10}(q^6-1)(q^4-1)(q^2-1)^2. \quad (4.37)$$

Since

$$4q^3 < \frac{v-1}{q^{10}(q^6-1)(q^4-1)(q^2-1)^2} < 14641a^2,$$

we conclude that $q \in \{2, 3, 4, 8, 9, 16, 27, 32\}$. For these values of q , if $m \geq 374$, then it is easy to check that $m(v-1) \geq 374(v-1) > 11^4 a^2 |X \cap H|$, and this violates (4.37). Therefore, $m \leq 373$, and so, for each such m and $q \in \{2, 3, 4, 8, 9, 16, 27, 32\}$, we see that (4.36) is not a positive integer, which is a contradiction.

(ii)-(iii) Our argument in these cases are similar, so we only deal with the case where $X \cap H = 2 \cdot \Omega_9(q)$ with q odd. Then $|X \cap H| = q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1)$, and so by (4.1), we have that $v = q^8(q^8 + q^4 + 1)$. As $|\text{Out}(X)| = 2a$, by Lemma 2.4(c), we conclude that

$$k \mid 2aq^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1). \quad (4.38)$$

Note that $\gcd((q^8-1)(q^6-1)(q^4-1)(q^2-1), v-1) = q^4 + 1$. Then applying Lemma 2.4(a) and Tit's lemma 2.2, k must divide $2\lambda a(q^4+1)$, and so $mk = 2\lambda a(q^4+1)$ for some positive integer m . Since $\lambda < k$, we have that

$$m < 2a(q^4+1). \quad (4.39)$$

Moreover, it follows from Lemma 2.4(a) that

$$\frac{2\lambda a(q^4+1)}{m}(k-1) = \lambda(q^{16} + q^{12} + q^8 - 1),$$

Therefore

$$k = \frac{m(q^{12} + q^4 - 1)}{2a} + 1, \quad (4.40)$$

$$4a^2\lambda = m^2(q^8 - q^4 + 2) - \frac{3m^2 - 2ma}{q^4 + 1} \quad (4.41)$$

Since λ is integer, $q^4 + 1$ divides $3m^2 - 2ma$, and so

$$m > \frac{q^2 + 1}{\sqrt{3}}. \quad (4.42)$$

We claim that $\gcd(k, q^{16}) < q^4$. Assume the contrary. Then q^4 divides $2ak$. Since $mq^4(q^8+1) - [m(q^{12}+q^4-1)+2a] = m-2a$, we have

$$mq^4(q^8+1) - 2ak = m - 2a,$$

and so q^4 must divide $m - 2a$. Thus $m - 2a = uq^4$ for some integer u . By (4.39),

$$u < 2a. \quad (4.43)$$

Recall that q^4+1 divides $3m^2-2ma = (3q^4u^2+10au-3u^2)(q^4+1)+(2a-u)(4a-3u)$. Then q^4+1 must divide $(2a-u)(4a-3u)$, and so (4.43) implies that $q^4+1 < 4a$, which is a contradiction. Therefore, $\gcd(k, q^{16}) < q^4$, as claimed.

TABLE 12. Some parameters for Lemma 4.9(ii)-(iii).

q	a	m	v	k	λ
3	1	28	43584723	7441295	1270465
3	1	82	43584723	21792362	10896181
9	2	772	1853302661435043	54508901806914	1603202994321
9	2	3604	1853302661435043	254469018279942	34940046551391
9	2	4376	1853302661435043	308977920086855	51512015326885
9	2	13124	1853302661435043	926651330717522	463325665358761

Now by (4.38), we have $k \mid 2af(q)$, where $f(q) = q^4(q^8 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1)$, and so (4.40) implies that

$$m(q^{12} + q^4 - 1) + 2a \mid 4a^2f(q), \quad (4.44)$$

Since

$$4a^2h(q)[m(q^{12} + q^4 - 1) + 2a] - ma^2f(q) = 4ma^2d(q) + 8a^3h(q)$$

where $h(q) = q^{12} - q^{10} - q^8 - q^4 + 3q^2 + 2$ and $d(q) = q^{10} + q^8 + 4q^6 + 2q^4 - 3q^2 - 2$, we conclude from (4.44) that $m(q^{12} + q^4 - 1) + 2a \leq 4ma^2d(q) + 8a^3h(q)$, and so (4.42) implies that

$$q^2 < \frac{(q^2 + 1)(q^{12} + q^4 - 4a^2d(q) - 1)}{h(q)} \leq 14a^3.$$

As in this case q is odd, this inequality implies that $q = 3$ or 9 , and so by (4.39), m is at most 164 or 26248, respectively. For each such value of q and m , the parameter k and λ obtained in (4.40) must be positive integers and all parameters must satisfy conditions of symmetric designs, and this leads us to the parameters in Table 12. However, by (4.38), k must divide 263139026617958400 or $88987349938389359442577906728960000$, respectively for $q = 3$ or 9 , which is a contradiction.

(iv) Suppose $d^2 \cdot D_4(q) \cdot \text{Sym}_3$ with $d = \gcd(2, q - 1)$. Then by (4.1), we have that $v = q^{12}(q^8 + q^4 + 1)(q^4 + 1)/6$. Here $|\text{Out}(X)|$ divides $2a$. Hence by Lemma 2.4(a) and Lemma 2.1(b) we have that

$$k \mid 12ag(q), \quad (4.45)$$

where $g(q) = q^{12}(q^6 - 1)(q^4 - 1)^2(q^2 - 1)$. Note that $\gcd((q^6 - 1)(q^4 - 1)^2(q^2 - 1), v - 1)$ divides $2 \cdot 12 \cdot 27a\lambda(q^2 - 1)^5(q^2 + 1)$. So there exists a positive integer m such that $mk = 648a\lambda(q^2 - 1)^5(q^2 + 1)$. Again by Lemma 2.4(a), we have that

$$k = \frac{mw(q)}{3888af(q)} + 1, \quad (4.46)$$

where $w(q) = q^{20} + 3q^{16} + 5q^{12} + 6q^8 + 6q^4 + 6$ and $f(q) = (q^2 - 1)^4$. This together with (4.45) implies that

$$mw(q) + 3888af(q) \mid 46656a^2f(q)g(q). \quad (4.47)$$

Note that

$$46656a^2h(q)[mw(q) + 3888a] - 46656ma^2g(q) = 46656ma^2d(q) + 3888 \cdot 46656a^3h(q),$$

where $h(q) = q^{18} - 4q^{16} + 19q^{12} - 15q^{10} - 40q^8 + 45q^6 + 55q^4 - 69q^2 - 65$ and $d(q) = 73q^{18} + 73q^{16} + 161q^{14} + 122q^{12} + 234q^{10} + 300q^8 + 3888q^6 + 60q^4 + 414q^2 + 390$. Then (4.47) implies that

$$mw(q) + 3888a \mid 46656ma^2d(q) + 3888 \cdot 46656a^3h(q),$$

and so $q^{20} + 3q^{16} + 5q^{12} + 6q^8 + 6q^4 + 6 = w(q) < mw(q) + 3888a \leq 46656[ma^2d(q) + 3888a^3h(q)] \leq 46656a^2[d(q) + 3888qh(q)]$. Thus

$$\left(\frac{1}{100}\right)q^2 < \frac{q^{20} + 3q^{16} + 5q^{12} + 6q^8 + 6q^4 + 6}{d(q) + 3888qh(q)} < 46656a^2.$$

This inequality holds only for (p, a) as in Table 13 below:

TABLE 13. Some parameters for Lemma 4.9(iv).

p	2	3	5	7	11	13, ..., 23	31, ..., 103	107, ..., 5701
$a \leq$	16	9	6	5	4	3	2	1

(v) Suppose $H \cap X = {}^3D_4(q) \cdot 3$. Then by (4.1), we have that $v = q^{12}(q^8 - 1)(q^4 - 1)/3$. Note that $|\text{Out}(X)|$ divides $2a$. Hence by Lemma 2.4(a) and Lemma 2.1(b) we have that

$$k \mid 6ag(q), \quad (4.48)$$

where $g(q) = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. Moreover, Lemma 2.4(a) implies that k divides $\lambda(v - 1)$. By Tit's lemma 2.2, $v - 1$ is coprime to q and also $\gcd((q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1), v - 1)$ divides $61(q^8 + q^4 + 1)$. Thus k must divide $366a\lambda(q^8 + q^4 + 1)$. So there exists a positive integer m such that $mk = 366a\lambda(q^8 + q^4 + 1)$. By Lemma 2.4(a), $k(k - 1) = \lambda(v - 1)$. Therefore

$$k = \frac{mw(q)}{1098a} + 1, \quad (4.49)$$

where $w(q) = q^{16} - 2q^{12} + 3q^4 - 3$. Now applying (4.48) and (4.49), we must have

$$mw(q) + 1098a \mid 6588a^2g(q). \quad (4.50)$$

Let now $h(q) = q^{12} - q^{10} + 3q^8 - 4q^6 + 8q^4 - 10q^2 + 14$ and $d(q) = 18q^{14} - 23q^{12} - 9q^{10} + 15q^8 - 18q^6 + 18q^4 + 30q^2 - 42$. Then

$$\begin{aligned} 6588a^2 \cdot h(q) \cdot [mw(q) + 1098a] \\ - 6588ma^2g(q) = 6588ma^2d(q) + 6588 \cdot 1098a^3h(q). \end{aligned}$$

Therefore (4.50) implies that

$$mw(q) + 1098a \leq 6588ma^2d(q) + 7233624a^3h(q),$$

and so

$$q^{16} - 2q^{12} + 3q^4 - 3 < 6588a^2[d(q) + 1098qh(q)]$$

Therefore

$$\frac{q^2}{578} < \frac{q^{16} - 2q^{12} + 3q^4 - 3}{d(q) + 1098qh(q)} \leq 6588a^2.$$

This inequality holds only for (p, a) as in Table 14.

TABLE 14. Some parameters for Lemma 4.9(v).

p	2	3	5	7	11, ..., 17	19, ..., 61	67, ..., 1951
$a \leq$	14	8	5	4	3	2	1

(vi) Suppose $Sp_4(q^2).2$ with $p = 2$. Then by (4.1), we have that

$$v = \frac{q^{16}(q^{12} - 1)(q^6 - 1)(q^2 - 1)}{2 \cdot (q^4 - 1)}.$$

Here $|\text{Out}(X)|$ divides $2a$, and so by Lemma 2.4(a) and 2.1(b), we have that

$$k \mid 4ag(q), \quad (4.51)$$

where $g(q) = q^8(q^8 - 1)(q^4 - 1)$. Since by Lemma 2.4(c), $\lambda v \leq k^2$, we must have

$$\lambda < \frac{4^2 a^2 g(q)^2}{v}. \quad (4.52)$$

Now Lemma 2.4(a) implies that k divides $\lambda(v - 1)$. By Tit's lemma 2.2, $v - 1$ is coprime to q and also $\gcd(v - 1, (q^4 - 1)(q^8 - 1))$ divides $25(q^4 + 1)$. Thus k must divide $100a\lambda(q^4 + 1)$. Again, by Lemma 2.4(c), as $\lambda v \leq k^2$, we conclude by (4.52) that

$$\begin{aligned} v < \frac{k^2}{\lambda} &\leq \frac{100^2 a^2 \lambda^2 (q^4 + 1)^2}{v \lambda} = \frac{100^2 a^2 \lambda (q^4 + 1)^2}{v} \\ &\leq \frac{4^2 \cdot 100^2 a^4 (q^4 + 1)^2 g(q)^2}{v}. \end{aligned}$$

Thus $v^2 \leq 4^2 \cdot 100^2 a^4 (q^4 + 1)^2 g(q)^2$, and so

$$\frac{q^{15}}{4} < \frac{q^{16}(q^8 + q^4 + 1)^2(q^4 + q^2 + 1)^2}{(q^4 + 1)^4(q^2 + 1)^2} < 4^3 \cdot 100^2 a^4.$$

Therefore, $q^{15} < 4^4 \cdot 100^2 a^4$, which is impossible for any $q = p^a$.

(vii) Suppose $X \cap H = Sp_4(q)^2 \cdot 2$ with $p = 2$. Here by (4.1), we have that

$$v = \frac{q^{16}(q^2 - 1)^2(q^8 + q^4 + 1)(q^4 + q^2 + 1)}{2}.$$

Note that in this case $p = 2$. Hence, by Lemmas 2.4(d) and 2.1, since $|\text{Out}(X)| = 2a$ we have $k \mid 4aq^8(q^4 - 1)^2(q^2 - 1)^2$. On the other hand, Lemma 2.4(a) implies that, k divides $\lambda(v - 1)$. By Tit's lemma 2.2, $v - 1$ is coprime to p and also $\gcd(v - 1, (q^2 - 1)^4(q^2 + 1)^2)$ divides 25. Thus $k \mid 25a\lambda$. Therefore there exists a positive integer m such that $mk = 25a\lambda$. As $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(v - 1)}{25a} + 1. \quad (4.53)$$

Applying (4.53), we have that

$$m(v - 1) + 25a \mid 100a^2 \cdot q^8(q^4 - 1)^2(q^2 - 1)^2. \quad (4.54)$$

Therefore

$$q^{11} \leq \frac{q^8(q^8 + q^4 + 1)(q^4 + q^2 + 1)(q^2 - 1)^2}{(q^4 - 1)^2(q^2 - 1)^2} < 100a^2,$$

and so $q^{11} < 100a^2$, which is impossible.

(viii) Suppose $X \cap H = {}^2F_4(q)$ with $q = 2^{2n+1} \geq 2$. Then by (4.1),

$$v = q^{12}(q^6 - 1)(q^4 + 1)(q^3 + 1)(q + 1).$$

As in this case q is even, $|\text{Out}(X)| = 2a$. Applying Lemmas 2.4(c) and 2.1(b), we observe that

$$k \mid 2ag(q), \quad (4.55)$$

where $q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$. Moreover, as $\gcd(v - 1, (q^3 + 1)(q^2 - 1)(q - 1)) = 1$, Lemma 2.4(a) and Tit's lemma 2.2 implies k divide $2a\lambda f(q)$, where $f(q) = (q^2 + 1)^2(q^4 - q^2 + 1)$. Therefore there exists a positive integer m such that $mk = 2a\lambda f(q)$. Since $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(v - 1)}{2a\lambda f(q)} + 1. \quad (4.56)$$

Now we apply (4.55) and (4.56), and conclude that

$$m(v - 1) + 2a\lambda f(q) \mid 4a^2 f(q)g(q). \quad (4.57)$$

Let now $h(q) = q^8 - 2q^7 + 3q^6 - 2q^5 - 2q^4 + 6q^3 - 5q^2 - 2q + 13$ and $d(q) = 16q^{25} + 12q^{24} - 20q^{23} - 4q^{22} + 16q^{21} - 24q^{19} - 20q^{18} + 12q^{17} + 8q^{16} - 12q^{15} - 8q^{14} + 12q^{13} + 12q^{12} - q^8 + 2q^7 - 3q^6 + 2q^5 + 2q^4 - 6q^3 + 5q^2 + 2q - 13$. Then

$$4a^2 h(q)[m(v - 1) + 2a\lambda f(q)] - 4ma^2 f(q)g(q) = 4ma^2 d(q) + 8a^3 f(q)h(q).$$

Therefore (4.57) implies that

$$m(v - 1) + a\lambda f(q) < 4a^2 m[d(q) + 2qf(q)h(q)],$$

and so

$$\frac{q}{16} < \frac{v - 1}{d(q) + 2qf(q)h(q)} < 4a^2.$$

and so $q < 64a^2$. Since $q = 2^{2n+1} \geq 2$, this inequality holds only for $q = 2^a$ where $a \leq 13$ is odd. We now apply Algorithm 1 but no parameter arise.

(ix) Suppose $H_0 = \text{Soc}(H) = PSL_2(q) \times G_2(q)$ with $q > 3$ odd. Note that $|\text{Out}(X)| = \gcd(2, p)a$ and $|H| = bq^7(q^2 - 1)^2(q^6 - 1)$, where b is a divisor of $|\text{Out}(H_0)|$ which divides $2 \cdot \gcd(2, q - 1)a^2$. Then (4.1) implies that

$$v = \frac{cq^{17}(q^4 + 1)(q^2 + 1)(q^{12} - 1)}{b},$$

where $c \mid 2a$ and $b \mid 4a^2$.

Note by Lemma 2.4(d) that k divides $bq^7(q^2 - 1)^2(q^6 - 1)$. Now Lemma 2.4(a) implies that k divides $\lambda(v - 1)$. By Tit's lemma 2.2, $v - 1$ is coprime to q , and so $\gcd(v - 1, (q^2 - 1)^2(q^6 - 1))$ divides b^7 . Thus $mk = b^8\lambda$ for some positive integer m . It follows from $k(k - 1) = \lambda(v - 1)$ that

$$k = \frac{m(v - 1)}{b^8} + 1 \quad (4.58)$$

Recall that $k \mid bq^7(q^2 - 1)^2(q^6 - 1)$. Then, by (4.96), we must have

$$m(v - 1) + b^8 \mid b^9(q^2 - 1)^2(q^6 - 1). \quad (4.59)$$

Since $m \geq 1$, $c \mid 2a$ and $b \mid 4a^2$, it follows from (4.97) that $q^{17}(q^4 + 1)(q^2 + 1)(q^{12} - 1) < 2^{20}a^{20}(q^2 - 1)^2(q^6 - 1)$, which is impossible. \square

Lemma 4.10. *If $X = E_6^\epsilon(q)$, then H is as in Table 2.*

Proof. Suppose $X = E_6^\epsilon(q)$. Then by Theorem 3.1, $X \cap H$ is one of the following:

- (i) $d \cdot (A_1(q) \times A_5^\epsilon(q)) \cdot d \cdot e$ with $d = \gcd(2, q-1)$ and $e = \gcd(3, q-\epsilon 1)$;
- (ii) $d^2 \cdot (D_4(q) \times (\frac{q-\epsilon 1}{d})^2) \cdot d^2 \cdot S_3$ with $d = \gcd(2, q-1)$;
- (iii) $({}^3D_4(q) \times (q^2 + \epsilon q + 1)) \cdot 3$;
- (iv) $h \cdot (D_5^\epsilon(q) \times (\frac{q-\epsilon 1}{h})) \cdot h$ with $h = \gcd(4, q-1)$;
- (v) $\text{Soc}(H) = C_4(q)$ with q odd;

(i) Suppose $X \cap H = d \cdot (A_1(q) \times A_5^\epsilon(q)) \cdot d \cdot e$. Then by (4.1) we have that

$$v = \frac{q^{20}(q^8 + q^4 + 1)(q^6 + \epsilon q^3 + 1)(q^4 + 1)(q^2 + 1)}{e}. \quad (4.60)$$

where $e = \gcd(3, q - \epsilon 1)$.

Let first $e = 1$. Since $|\text{Out}(X)| = 2a$, Lemmas 2.4(a) and 2.1(b) imply that

$$k \mid ag(q), \quad (4.61)$$

where $g(q) = q^{16}(q^6 - 1)(q^5 - \epsilon 1)(q^4 - 1)(q^3 - \epsilon 1)(q^2 - 1)^2$. Moreover, it follows from Lemmas 2.4(a) and 2.2 that k divides $\gcd(v-1, g(q))$, and this divides $f(q)$, where $f(q) = (q^5 - \epsilon 1)(q^2 - 1)^4(q - \epsilon 1)$. Thus k must divide $2a\lambda f(q)$, and so $mk = 2a\lambda f(q)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{2af(q)} + 1. \quad (4.62)$$

Now we apply (4.61), and so

$$m(v-1) + 2af(q) \mid 4a^2f(q)g(q) \quad (4.63)$$

Then $4a^2h(q)[m(v-1) + 2af(q)] - ma^2f(q)g(q) = 4ma^2d(q) + \epsilon 8a^3f(q)h(q)$, where

$$\begin{aligned} h(q) &= q^{12} - \epsilon q^{11} - 7q^{10} + \epsilon 5q^9 + 21q^8 - \epsilon 7q^7 - 33q^6 - \epsilon 3q^5 + 23q^4 + \epsilon 8q^3 + q^2 + \epsilon 23q + 13, \\ d(q) &= 49q^{39} + \epsilon 67q^{38} + 52q^{37} + \epsilon 62q^{36} + 106q^{35} + \epsilon 91q^{34} + 104q^{33} + \epsilon 127q^{32} + \\ &\quad 143q^{31} + \epsilon 121q^{30} + 138q^{29} + \epsilon 129q^{28} + 90q^{27} + \epsilon 60q^{26} + 86q^{25} + \epsilon 63q^{24} + \\ &\quad 31q^{23} + \epsilon 33q^{22} + 33q^{21} - \epsilon 2q^{20} - 5q^{19} + \epsilon 6q^{18} + q^{17} - \epsilon q^{16} - \epsilon q^{12} + q^{11} + \\ &\quad \epsilon 7q^{10} - 5q^9 - \epsilon 21q^8 + 7q^7 + \epsilon 33q^6 + 3q^5 - \epsilon 23q^4 - 8q^3 - \epsilon q^2 - 23q - \epsilon 13. \end{aligned}$$

Thus by (4.63), we conclude that $m(v-1) + 2af(q)$ divides $4ma^2d(q) + \epsilon 8a^3f(q)h(q)$, and so $m(v-1) + 2af(q) < 4ma^2d(q) + \epsilon 8a^3f(q)h(q) < 4ma^2[d(q) + \epsilon 2qf(q)h(q)]$, and hence

$$\frac{q}{u_\epsilon} < \frac{v}{d(q) + \epsilon 2qf(q)h(q)} < 4a^2, \quad (4.64)$$

where $u_+ = 49$ and $u_- = 73$. This inequality holds if (p, a, ϵ) is as in Table 15.

Let now $e = 3$. Then $|\text{Out}(X)| = 6a$, and so by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid 6ag(q), \quad (4.65)$$

where $g(q) = q^{16}(q^6 - 1)(q^5 - \epsilon 1)(q^4 - 1)(q^3 - \epsilon 1)(q^2 - 1)^2$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v-1, g(q))$, and this divides $81f(q)$, where $f(q) = (q - \epsilon 1)^6$. Thus k must divide $81a\lambda f(q)$, and so $mk = 81a\lambda f(q)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, we conclude that

$$k = \frac{m(v-1)}{6af(q)} + 1. \quad (4.66)$$

TABLE 15. Possible value for p and a for Lemma 4.10(i) when $\gcd(3, q - \epsilon 1) = 1$.

$\epsilon = +$		$\epsilon = -$	
p	$a \leq$	p	$a \leq$
2	15	2	16
3	8	3	9
5	5	5	5
7	4	7	4
11	3	11, 13	3
$13 \leq p \leq 23$	2	$17 \leq p \leq 31$	2
$29 \leq p \leq 193$	1	$37 \leq p \leq 283$	1

Now we apply (4.65), and so

$$m(v-1) + 6af(q) \mid 36a^2 f(q)g(q) \quad (4.67)$$

Then $ma^2 f(q)g(q) - 36a^2 h(q)[m(v-1) + 6af(q)] = 36a^2 [\epsilon md(q) - 6af(q)h(q)]$, where

$$\begin{aligned} h(q) &= 3q^{12} - \epsilon 3q^{11} - 21q^{10} + \epsilon 15q^9 + 63q^8 - \epsilon 21q^7 - 99q^6 - \epsilon 9q^5 + 69q^4 + \epsilon 24q^3 + 3q^2 + \epsilon 69q + 39 \\ d(q) &= 49q^{39} + \epsilon 67q^{38} + 52q^{37} + \epsilon 62q^{36} + 106q^{35} + \epsilon 91q^{34} + 104q^{33} + \epsilon 127q^{32} + 143q^{31} + \epsilon 121q^{30} + 138q^{29} + \epsilon 129q^{28} + 90q^{27} + \epsilon 60q^{26} + 86q^{25} + \epsilon 63q^{24} + 31q^{23} + \epsilon 33q^{22} + 33q^{21} - \epsilon 2q^{20} - 5q^{19} + \epsilon 6q^{18} + q^{17} - \epsilon q^{16} - \epsilon 3q^{12} + 3q^{11} + \epsilon 21q^{10} - 15q^9 - \epsilon 63q^8 + 21q^7 + \epsilon 99q^6 + 9q^5 - \epsilon 69q^4 - 24q^3 - \epsilon 3q^2 - 69q - \epsilon 39 \end{aligned}$$

Thus by (4.63), we conclude that $m(v-1) + 6af(q)$ divides $36a^2 |\epsilon md(q) - 6f(q)h(q)a|$, and so

$$\frac{q}{u_\epsilon} < \frac{v}{d(q) + \epsilon 6qf(q)h(q)} < 36a^2, \quad (4.68)$$

where $u_+ = 217$ and $u_- = 147$, and this inequality if (p, a, ϵ) as in Table 16.

TABLE 16. Possible value for p and a for Lemma 4.10(i) when $\gcd(3, q - \epsilon 1) = 3$.

$\epsilon = +$		$\epsilon = -$	
p	$a \leq$	p	$a \leq$
2	21	2	21
3	12	3	12
5	7	5	8
7	6	7	6
11, 13, 17	4	11	5
19, ..., 41	3	13, 17	4
43, ..., 173	2	19, ..., 31	3
179, ..., 7793	1	37, ..., 139	2
		149, ..., 5281	1

(ii) Suppose $X \cap H = d^2 \cdot (D_4(q) \times (\frac{q-\epsilon 1}{d})^2) \cdot d^2 \cdot S_3$ with $d = \gcd(2, q-1)$. Then by (4.1), we have that

$$v = \frac{q^{24}(q^{12}-1)(q^9-\epsilon 1)(q^8-1)(q^5-\epsilon 1)}{6e(q^4-1)^2(q-\epsilon 1)^2}, \quad (4.69)$$

where $e = \gcd(3, q-\epsilon 1)$. Note that $|\text{Out}(X)| = 2ea$. So by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid 12eag(q), \quad (4.70)$$

where $g(q) = q^{12}(q^6-1)(q^4-1)^2(q^2-1)(q-\epsilon 1)^2$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v-1, g(q))$, and this divides $f(q)$, where

$$f(q) = \begin{cases} (q-\epsilon 1)^6(q+\epsilon 1)^4(q^2+1)^2, & \text{if } e = 1; \\ 2^6(q-\epsilon 1)^6, & \text{if } e = 3. \end{cases}$$

Thus $mk = 12ea\lambda f(q)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{12eaf(q)} + 1, \quad (4.71)$$

and so (4.70) implies that

$$m(v-1) + 12eaf(q) \mid 144e^2a^2f(q)g(q). \quad (4.72)$$

Thus $v-1 \leq m(v-1) + 12eaf(q) < 144e^2a^2f(q)g(q)$.

If $e = 1$, then since $q^4 < 8(v-1)/f(q)g(q)$, it follows that $q^8 < 8 \cdot 144a^2$ which implies that $q = 4$. If $e = 3$, then since $q^4 < u_p(v-1)/f(q)g(q)$ with $u_2 = 144$ and $u_p = 3$ for $p \neq 2$, we conclude that $q^8 < 9 \cdot 144u_pa^2$, and hence $q \in \{2, 5, 8, 32\}$. In conclusion, $q \in \{2, 4, 5, 8, 32\}$ for which we obtain v and k by (4.69) and (4.73), and then observe that $k(k-1) = \lambda(v-1)$ does not hold, for any positive integer λ .

(iii) Suppose $X \cap H = ({}^3D_4(q) \times (q^2 + \epsilon q + 1)) \cdot 3$. Then by (4.1) we have that

$$v = \frac{q^{24}(q^9-1)(q^8-1)(q^5-1)(q^4-1)}{3e \cdot (q^2 + \epsilon q + 1)}.$$

where $e = \gcd(3, q-\epsilon 1)$. As $|\text{Out}(X)|$ divides $2ea$, it follows by Lemma 2.4(a) and Lemma 2.1(b) that

$$k \mid 6eag(q), \quad (4.73)$$

where $g(q) = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)(q^2+q+1)$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $6ea\lambda \cdot \gcd(v-1, g(q))$, and hence k divides $6ea\lambda f(q)$, where $f(q) = (q^8+q^4+1)(q^4+q^2+1)(q^2+\epsilon q+1)$. So $mk = 16ea\lambda f(q)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{6eaf(q)} + 1, \quad (4.74)$$

and so by (4.73), we conclude that

$$m(v-1) + 6eaf(q) \mid 36e^2a^2f(q)g(q). \quad (4.75)$$

Therefore, $q^4 < v/f(q)g(q) \leq 36e^2a^2$ which implies that $q = 2$. This case can be ruled out by Lemmas 2.4 and 2.6.

(iv) Suppose $X \cap H = h \cdot (D_5^\epsilon(q) \times (\frac{q-\epsilon 1}{h})) \cdot h$ with $h = \gcd(4, q - \epsilon 1)$. Then by (4.1) we must have that

$$v = \frac{q^{16}(q^{12} - 1)(q^9 - \epsilon 1)}{e \cdot (q^4 - 1)(q - \epsilon 1)}.$$

where $e = \gcd(3, q - 1)$. Note that $|\text{Out}(X)| = 2ea$. So by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid 2eag(q), \quad (4.76)$$

where $g(q) = q^{20}(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^4 - 1)(q^2 - 1)(q - \epsilon 1)$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v - 1, g(q))$, and this divides $f(q)$, where

$$f(q) = \begin{cases} 13^6(7q - \epsilon 5)(3q + \epsilon 2)(q^4 + 1), & \text{if } e = 1; \\ 3^2 \cdot 6^7(3q - \epsilon 13)(q + \epsilon 1)^2(q^2 + 1)^2, & \text{if } e = 3. \end{cases}$$

Thus $mk = 12ea\lambda f(q)$, for some positive integer m . Since $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(v - 1)}{2eaf(q)} + 1, \quad (4.77)$$

and so (4.76) implies that

$$m(v - 1) + 2eaf(q) \mid 4e^2a^2f(q)g(q). \quad (4.78)$$

Then $me^2a^2f(q)g(q) - 4eah_e(q)[m(v - 1) + 2eaf(q)] = 4e^2a^2[\epsilon md_e(q) - 2af(q)h_e(q)]$, where

$$\begin{aligned} h_1(q) &= 917093710q^{31} + \epsilon 743328586q^{30} + 188245551q^{29} + \epsilon 584043889q^{28} + \\ &1544578880q^{27} + \epsilon 1221182677q^{26} + 772289440q^{25} + \epsilon 1819706993q^{24} + \\ &1689383150q^{23} + \epsilon 1221182677q^{22} + 1076378407q^{21} + \epsilon 1906589555q^{20} + \\ &724021350q^{19} + \epsilon 424759192q^{18} + 1042590744q^{17} + \epsilon 1008803081q^{16} + \\ &14480427q^{15} + \epsilon 193072360q^{14} - 53094899q^{13} - \epsilon 352357057q^{12} + \\ &521295372q^{11} - \epsilon 82055753q^{10} - 738501777q^9 + \epsilon 830211148q^8 - \\ &299262158q^7 - \epsilon 313742585q^6 + 342703439q^5 + \epsilon 299262158q^4 - \\ &468200473q^3 + \epsilon 279954922q^2 + 477854091q - \epsilon 1221182677, \\ d_1(q) &= 101362989q^{20} - \epsilon 207552787q^{19} - 38614472q^{18} + \epsilon 299262158q^{17} - \\ &212379596q^{16} + \epsilon 14480427q^{15} + 193072360q^{14} - \epsilon 53094899q^{13} - \\ &352357057q^{12} + \epsilon 521295372q^{11} - 82055753q^{10} - \epsilon 738501777q^9 + \\ &830211148q^8 - \epsilon 299262158q^7 - 313742585q^6 + \epsilon 342703439q^5 + \\ &299262158q^4 - \epsilon 468200473q^3 + 279954922q^2 + \epsilon 477854091q - 1221182677, \\ h_3(q) &= 108335232q^{31} - \epsilon 105815808q^{30} - 559312128q^{29} - \epsilon 624817152q^{28} - \\ &942264576q^{27} - \epsilon 1254673152q^{26} - 1320178176q^{25} - \epsilon 1335294720q^{24} - \\ &1237037184q^{23} - \epsilon 1322697600q^{22} - 1184129280q^{21} - \epsilon 783540864q^{20} - \\ &869201280q^{19} - \epsilon 793618560q^{18} - 216670464q^{17} - \epsilon 559312128q^{16} - \\ &166281984q^{15} + \epsilon 272097792q^{14} - 37791360q^{13} + \epsilon 476171136q^{12} + \\ &513962496q^{11} - \epsilon 7558272q^{10} - 325005696q^9 - \epsilon 68024448q^8 - 823851648q^7 - \\ &\epsilon 778502016q^6 + 461054592q^5 - \epsilon 498845952q^4 + 589545216q^3 + \\ &\epsilon 1163973888q^2 + 355238784q + \epsilon 566870400, \end{aligned}$$

$$\begin{aligned}
 d_3(q) = & 22674816q^{21} - \epsilon 98257536q^{20} - 22674816q^{19} + \epsilon 98257536q^{18} - 90699264q^{17} + \\
 & \epsilon 370355328q^{16} + 166281984q^{15} - \epsilon 272097792q^{14} + 37791360q^{13} - \\
 & \epsilon 476171136q^{12} - 513962496q^{11} + \epsilon 7558272q^{10} + 325005696q^9 + \\
 & \epsilon 68024448q^8 + 823851648q^7 + \epsilon 778502016q^6 - 461054592q^5 + \\
 & \epsilon 498845952q^4 - 589545216q^3 - \epsilon 1163973888q^2 - 355238784q - \epsilon 566870400.
 \end{aligned}$$

Thus by (4.78), we conclude that $m(v-1)+2af(q)$ divides $4e^2a^2|md(q)-2\epsilon af(q)h(q)|$, and so

$$v + 2af(q) < 4e^2a^2|d(q) - 2\epsilon af(q)h(q)|. \quad (4.79)$$

This inequality holds if (p, a, ϵ) as in Table 17.

TABLE 17. Possible values for p and a in Lemma 4.10(iv).

$\gcd(3, q - \epsilon 1) = 1$		$\gcd(3, q - \epsilon 1) = 3$	
p	$a \leq$	p	$a \leq$
2	95	2	105
3	54	3	60
5	33	5	37
7	25	7	29
11	19	11	22
13	17	13	20
17	15	17	18
19	14	19	17
23	13	23	15
29	12	29	14
31, 37	11	31, 37	13
41, 43, 47	10	41, 43	12
53, 59, 61	9	47, 53, 59	11
67, ..., 89	8	61, ..., 79	10
97, ..., 137	7	83, ..., 113	9
139, ..., 229	6	127, ..., 167	8
233, ..., 443	5	173, ..., 281	7
449, ..., 1051	4	283, ..., 523	6
1061, ..., 3559	3	541, ..., 1181	5
3571, ..., 21089	2	1187, ..., 3533	4
21101, ..., 196597	1	3539, ..., 16963	3
		16979, ..., 197273	2
		197279, ..., 11943929	1

(v) Suppose that the socle H_0 of H is $C_4(q)$ with q odd. Then $|H| = b|H_0|$, where b divides $(2, q-1)a = 2a$. Note that $|G| = c|X|$ for some divisor c of $2ea$, where $e = \gcd(3, q - \epsilon 1)$. Then by (4.1), we have that

$$v = \frac{cq^{20}(q^5 - \epsilon 1)(q^9 - \epsilon 1)(q^{12} - 1)}{eb(q^4 - 1)},$$

where $e = \gcd(3, q - \epsilon 1)$, $c \mid 2ea$ and $b \mid 2a$. So by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid bg(q), \quad (4.80)$$

where $g(q) = q^{16}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v-1, g(q))$, and this divides $f(q, e, c, b)$, where $f(q, e, c, b) = b^6 e^6 (be - 12c)^4 (6cq + ebe)^2 (cq^2 - be - c)$. Thus $mk = b\lambda f(q, e, c, b)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{bf(q, e, c, b)} + 1. \quad (4.81)$$

Since also $b \mid 2a$, by (4.80), we conclude that

$$m(v-1) + bf(q, e, c, b) \mid b^2 f(q, e, c, b)g(q). \quad (4.82)$$

Note that

$$e^5(6q + l_\epsilon e)^2(q^2 - be - c) \leq f(q, e, c, b) \leq e^5 b^5 (be + 12c)^4 (6cq + u_\epsilon e)^2 (cq^2 - e - 1), \quad (4.83)$$

where

$$u_\epsilon := \begin{cases} b, & \text{if } \epsilon = +; \\ -1, & \text{if } \epsilon = -. \end{cases} \quad l_\epsilon := \begin{cases} 1, & \text{if } \epsilon = +; \\ -b, & \text{if } \epsilon = -. \end{cases}$$

Since now $m \geq 1$, $b \mid 4a^2$ and $c \mid a$, it follows from (4.82) that $(v-1) + f(q, e, c, b) \leq b^2 f(q, e, c, b)g(q)$, and so (4.83) implies that

$$(v-1) + e^5(6q + l_\epsilon e)^2(q^2 - 2ae - a) < 4e^5 a^7 (2ae + 12a)^4 (6aq + u_\epsilon e)^2 (aq^2 - e - 1).$$

This inequality holds only for $q = p^a$ as in Table 18. \square

TABLE 18. Possible values for p and a in Lemma 4.10(iv).

$\gcd(3, q - \epsilon 1) = 1$		$\epsilon = -$ and $\gcd(3, q + 1) = 3$		$\epsilon = +$ and $\gcd(3, q - 1) = 3$	
p	$a \leq$	p	$a \leq$	p	$a \leq$
3	22	3	29	3	30
5	13	5	18	5	18
7	10	7	14	7	14
11	8	11	11	11	11
13	7	13	10	13	10
17, 19	6	17, 19	8	17	9
23, 29, 31	5	23	7	19, 23	8
37, ..., 53	4	29, ..., 41	6	37, ..., 53	6
59, ..., 131	3	43, ..., 67	5	59, ..., 89	5
137, ..., 619	2	71, ..., 127	4	97, ..., 223	4
631, ..., 18803	1	131, ..., 283	3	227, ..., 827	3
		293, ..., 279919	1	829, ..., 8461	2
				8467, ..., 2519423	1

Lemma 4.11. *If X is $E_7(q)$, then H is as in Table 2.*

Proof. Suppose $X = E_7(q)$. Then by Theorem 3.1, $X \cap H$ is one of the following:

- (i) $d \cdot (A_1(q) \times D_6(q)) \cdot d$ with $d = \gcd(2, q-1)$;
- (ii) $h \cdot (A_7^\epsilon(q) \cdot g \cdot (2 \times (2/h)))$ with $h = \gcd(4, q-\epsilon)/\gcd(2, q-1)$ and $g = \gcd(8, q-\epsilon)/\gcd(2, q-1)$;
- (iii) $e(E_6^\epsilon(q) \times (q - \epsilon/e)) \cdot e \cdot 2$ where $e = \gcd(3, q - \epsilon 1)$;
- (iv) $\text{Soc}(H) = PSL_2(q) \times F_4(q)$ with $q > 3$.

(i) Suppose $X \cap H = d \cdot (A_1(q) \times D_6(q)) \cdot d$ with $d = \gcd(2, q-1)$. Then by (4.1) we have that

$$v = \frac{q^{32}(q^{18}-1)(q^{14}-1)(q^6+1)}{(q^4-1)(q^2-1)}. \quad (4.84)$$

Here $|\text{Out}(X)| = da$. Then Lemmas 2.4(a) and 2.1(b) imply that

$$k \mid ag(q), \quad (4.85)$$

where $g(q) = q^{31}(q^{10}-1)(q^8-1)(q^6-1)^2(q^4-1)(q^2-1)^2$. Now we apply Lemmas 2.4(a) and 2.2, and conclude that k divides $\gcd(v-1, (q^{10}-1)(q^8-1)(q^6-1)^2(q^4-1)(q^2-1)^2)$ which is a divisor of $4f(q)$, where $f(q) = (q^{10}-1)(q^4+1)(q^2-1)^6$. Thus k must divide $4a\lambda f(q)$, and hence there exists a positive integer m such that $mk = 4a\lambda f(q)$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{4af(q)} + 1. \quad (4.86)$$

Note by (4.85) that $k \mid ag(q)$. Then, by (4.86), we must have

$$m(v-1) + 4af(q) \mid 4a^2f(q)g(q). \quad (4.87)$$

Let now

$$\begin{aligned} h(q) &= q^{31} - 9q^{29} + 35q^{27} - 78q^{25} + 115q^{23} - 131q^{21} + 135q^{19} - 129q^{17} + 105q^{15} - \\ &\quad 52q^{13} - 50q^{11} + 179q^9 - 288q^7 + 386q^5 - 471q^3 + 507q \\ d(q) &= 480q^{63} + 137q^{61} + 724q^{59} + 985q^{57} + 870q^{55} + 1849q^{53} + 1063q^{51} + 2101q^{49} + \\ &\quad 1398q^{47} + 1719q^{45} + 1261q^{43} + 1376q^{41} + 593q^{39} + 987q^{37} + 8q^{35} + 515q^{33} - \\ &\quad 2q^{31} + 9q^{29} - 35q^{27} + 78q^{25} - 115q^{23} + 131q^{21} - 135q^{19} + 129q^{17} - 105q^{15} + \\ &\quad 52q^{13} + 50q^{11} - 179q^9 + 288q^7 - 386q^5 + 471q^3 - 507q \end{aligned}$$

Then

$$4a^2h(q)[m(v-1) + af(q)] - 4ma^2f(q)g(q) = 4ma^2d(q) + 16a^3f(q)h(q),$$

and so (4.87) implies that

$$m(v-1) + af(q) \leq 4ma^2d(q) + 16a^3f(q)h(q) < 4ma^2[d(q) + 4qf(q)h(q)]$$

Note that $d(q) + 4qf(q)h(q) < 600q^{63}$. Thus

$$\frac{q^{32}(q^{18}-1)(q^{14}-1)(q^6+1)}{q^{63}(q^4-1)(q^2-1)} < 2400a^2 \quad (4.88)$$

This inequality only for $q = p^a$ as in Table 19.

TABLE 19. Possible values for p and a in Lemma 4.11(i).

p	2	3	5	7	11, 13	17, 19, 23	29, ..., 97	101, ..., 2399
$a \leq$	19	11	7	5	4	3	2	1

(ii) Suppose $X \cap H = h \cdot (A_7^\epsilon(q) \cdot g \cdot (2 \times (2/h)))$ where $h = \gcd(4, q-\epsilon 1) / \gcd(2, q-1)$, $g = \gcd(8, q-\epsilon 1) / \gcd(2, q-1)$ and $\epsilon = \pm$. Then by (4.1) we have that

$$v = \frac{q^{35}(q^{18}-1)(q^{12}-1)(q^7+\epsilon 1)(q^5+\epsilon 1)}{4 \cdot (q^4-1)(q^3-\epsilon 1)}.$$

Here $|\text{Out}(X)| = \gcd(2, q-1)a$. So by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid 4ag(q), \quad (4.89)$$

where $g(q) = q^{28}(q^8 - 1)(q^7 - \epsilon 1)(q^6 - 1)(q^5 - \epsilon 1)(q^4 - 1)(q^3 - \epsilon 1)(q^2 - 1)$. Since also by Lemma 2.4(a), k divides $\lambda(v - 1)$, Tit's lemma 2.2 implies that k divides $\gcd(v - 1, g(q)/q^{28})$ which is a divisor of $(q^8 - 1)(q^3 - \epsilon 1)^2(q^2 + 1)(q - 1)^4$. Thus k must divides $4a\lambda f(q)$, where $f(q) = (q^8 - 1)(q^3 - \epsilon 1)^2(q^2 + 1)(q - 1)^4$, so there exists a positive integer m such that $mk = 4a\lambda f(q)$. Since $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(v - 1)}{4a\lambda f(q)} + 1. \quad (4.90)$$

Note by (4.89) that $k \mid 4a\lambda g(q)$. Then, by (4.90), we must have

$$m(v - 1) + 4a\lambda f(q) \mid 16a^2\lambda f(q)g(q). \quad (4.91)$$

Then

$$16a^2h_\epsilon(q)[m(v - 1) + 4a\lambda f(q)] - 16ma^2\lambda f(q)g(q) = 16ma^2d_\epsilon(q) + 64a^3\lambda f(q)h_\epsilon(q),$$

where $\epsilon = \pm$ and

$$\begin{aligned} h_+(q) &= 4q^{13} - 16q^{12} + 24q^{11} - 32q^{10} + 56q^9 - 56q^8 + 44q^7 - 56q^6 + 16q^5 + 32q^4 - \\ &\quad 40q^3 + 88q^2 - 128q + 136, \\ h_-(q) &= 4q^{13} + 16q^{12} + 24q^{11} + 24q^{10} + 24q^9 - 44q^7 - 80q^6 - 128q^5 - 160q^4 - 128q^3 - \\ &\quad 48q^2 + 64q + 208, \\ d_\epsilon(q) &= 30q^{69} - \epsilon 30q^{68} + 24q^{67} + \epsilon 30q^{66} + 3q^{65} + \epsilon 36q^{64} - 6q^{63} + \epsilon 77q^{62} - q^{61} + \\ &\quad \epsilon 59q^{60} + 26q^{59} + \epsilon 78q^{58} + 48q^{57} + \epsilon 39q^{56} + 81q^{55} + \epsilon 58q^{54} + 104q^{53} - \epsilon 5q^{52} + \\ &\quad 112q^{51} + \epsilon 34q^{50} + 106q^{49} + \epsilon 9q^{48} + 78q^{47} + \epsilon 50q^{46} + 39q^{45} + \epsilon 60q^{44} - 6q^{43} + \\ &\quad \epsilon 94q^{42} - 37q^{41} + \epsilon 64q^{40} - 18q^{39} + \epsilon 47q^{38} + 5q^{37} - \epsilon 28q^{36} + 38q^{35} - \epsilon 6q^{34} + \\ &\quad 11q^{33} - \epsilon 11q^{32} + 7q^{31} - \epsilon 6q^{30} + 4q^{29} - \epsilon q^{28} - 4q^{13} + \epsilon 16q^{12} - 24q^{11} + \epsilon 32q^{10} - \\ &\quad 56q^9 + \epsilon 56q^8 - 44q^7 + \epsilon 56q^6 - 16q^5 - \epsilon 32q^4 + 40q^3 - \epsilon 88q^2 + 128q - \epsilon 136. \end{aligned}$$

Therefore (4.91) implies that

$$\begin{aligned} m(v - 1) + 4a\lambda f(q) &\leq 16ma^2d_\epsilon + 64a^3\lambda f(q)h_\epsilon(q) \\ &< 16ma^2(d_\epsilon(q) + 4qf(q)h_\epsilon(q)) \\ &< 16ma^2u_\epsilon q^{69}, \end{aligned}$$

where $u_+ = 30$ and $u_- = 50$. This implies that $v < 16ma^2u_\epsilon q^{69}$, where $u_+ = 30$ and $u_- = 50$, and so p and a are as in Table 20.

TABLE 20. Possible values for p and a for Lemma 4.11.

$\epsilon = +$		$\epsilon = -$	
p	$a \leq$	p	$a \leq$
2	17	2	17
3	9	3	10
5	6	5	6
7	4	7	5
11, 13	3	11, ..., 19	3
17, ..., 43	2	23, ..., 53	2
47, ..., 479	1	59, ..., 797	1

(iii) Suppose $X \cap H = e(E_6^\epsilon(q) \times (q - \epsilon/e)) \cdot e \cdot 2$ where $e = \gcd(3, q - \epsilon 1)$. Then (4.1) implies that

$$v = \frac{q^{27}(q^5 + \epsilon 1)(q^9 + \epsilon 1)(q^{14} - 1)}{2d(eq - \epsilon)},$$

where $e = \gcd(3, q - \epsilon 1)$ and $d = \gcd(2, q - 1)$. We easily observe that $e = 1$. Then

$$v = \frac{q^{27}(q^5 + \epsilon 1)(q^9 + \epsilon 1)(q^{14} - 1)}{2d(q - \epsilon)},$$

where $\gcd(3, q - \epsilon 1) = 1$ and $d = \gcd(2, q - 1)$. Since $\text{Out}(X) = da$ with $d = \gcd(2, q - 1)$, we conclude by Lemmas 2.4(a) and 2.1(b) that

$$k \mid adg(q), \quad (4.92)$$

where $g(q) = 2q^{36}(q - \epsilon 1)(q^2 - 1)(q^5 - \epsilon 1)(q^6 - 1)(q^8 - 1)(q^9 - \epsilon 1)(q^{12} - 1)$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v - 1, g(q))$, and this divides $f(q)$, where

$$f_\epsilon(q) = \begin{cases} 2^{14}3^7(q + 2)(q^2 + 2q - 1)(q^2 + \epsilon q + 1)(q^6 + \epsilon q^3 + 1), & \text{if } q \text{ even;} \\ 13^2(q - \epsilon 1)^7(q + \epsilon 3)(q^4 + 1), & \text{if } q \text{ odd.} \end{cases} \quad (4.93)$$

Thus k must divide $ad\lambda f_\epsilon(q)$, and so $mk = ad\lambda f_\epsilon(q)$, for some positive integer m . Since $k(k - 1) = \lambda(v - 1)$, it follows that

$$k = \frac{m(v - 1)}{ad f_\epsilon(q)} + 1. \quad (4.94)$$

Now we apply (4.92), and so

$$m(v - 1) + ad f_\epsilon(q) \mid a^2 d^2 f_\epsilon(q) g(q). \quad (4.95)$$

Then $ma^2 d^2 f_\epsilon(q) g(q) = a^2 d^2 h_\epsilon(q) [m(v - 1) + ad f_\epsilon(q)] + a^2 d^2 [md_\epsilon(q) + ad f_\epsilon(q) h_\epsilon(q)]$, where $\epsilon = \pm$ and

$$\begin{aligned} d_\epsilon(q) = & 13329432576q^{53} + 9889579008q^{51} + 28450455552q^{49} + 24652283904q^{47} + \\ & 33323581440q^{45} + 51884457984q^{43} + 33968553984q^{41} + 43069833216q^{39} + \\ & 48874586112q^{37} + 29382082560q^{35} + 29382082560q^{33} + 26945519616q^{31} + \\ & 1576599552q^{29} + 27303837696q^{27} - 3009871872q^{25} + 5016453120q^{23} - \\ & 7309688832q^{21} + 10032906240q^{19} - 6306398208q^{17} + 1146617856q^{15} + \\ & 6306398208q^{13} - 19635830784q^{11} + 27518828544q^9 - 34398535680q^7 + \\ & 34685190144q^5 - 19349176320q^3 + 429981696q + \epsilon(36691771392q^{52} - \\ & 5088116736q^{50} + 38841679872q^{48} + 17485922304q^{46} + 46008041472q^{44} + \\ & 32391954432q^{42} + 55252647936q^{40} + 25798901760q^{38} + 58477510656q^{36} + \\ & 20925775872q^{34} + 36333453312q^{32} + 12684460032q^{30} + 24078974976q^{28} + \\ & 2579890176q^{26} - 3439853568q^{24} + 1003290624q^{22} + 4013162496q^{20} - \\ & 7166361600q^{18} + 13902741504q^{16} - 18919194624q^{14} + 18059231232q^{12} - \\ & 14906032128q^{10} + 1003290624q^8 + 16625958912q^6 - 31961972736q^4 + \\ & 49591222272q^2 - 53461057536), \end{aligned}$$

$$\begin{aligned}
h_\epsilon(q) = & 143327232q^{36} - 286654464q^{34} + 429981696q^{32} - 1289945088q^{30} + \\
& 2436562944q^{28} - 2579890176q^{26} + 3439853568q^{24} - 1003290624q^{22} - \\
& 4013162496q^{20} + 7166361600q^{18} - 13902741504q^{16} + 18919194624q^{14} - \\
& 18059231232q^{12} + 14906032128q^{10} - 1003290624q^8 - 16625958912q^6 + \\
& 31961972736q^4 - 49591222272q^2 + 53461057536 + \epsilon(429981696q^{35} - \\
& 1146617856q^{33} + 429981696q^{31} + 143327232q^{29} - 573308928q^{27} + \\
& 3009871872q^{25} - 5016453120q^{23} + 7309688832q^{21} - 10032906240q^{19} + \\
& 6306398208q^{17} - 1146617856q^{15} - 6306398208q^{13} + 19635830784q^{11} - \\
& 27518828544q^9 + 34398535680q^7 - 34685190144q^5 + 19349176320q^3 - \\
& 429981696q),
\end{aligned}$$

Therefore (4.95) implies that $v - 1 < a^2 d^2 [d_\epsilon(q) + \text{adf}_\epsilon(q) h_\epsilon(q)]$, and this is true if $q = p^a$ is as in Table 21.

TABLE 21. Possible values for p and a in Lemma 4.11(iii).

$\gcd(3, q - \epsilon 1) = 1$		
p	$a \leq$	Conditions
2	45	
3	3	
5	2	
7, ..., 23	1	$\epsilon = +$
7, ..., 29	1	$\epsilon = -$

Note that $\gcd(q + 1, 3) \neq 1$ for q even. It is not also difficult to check that for odd values of $q = p^a$, $k(k - 1) = \lambda(v - 1)$ does not hold for any positive integer λ .

(iv) Suppose $H_0 = \text{Soc}(H) = \text{PSL}_2(q) \times F_4(q)$ with $q > 3$. Note that $|\text{Out}(X)| = \gcd(2, q - 1)a$ and $|H| = bq^{25}(q^2 - 1)^2(q^6 - 1)(q^8 - 1)(q^{12} - 1)$, where b is a divisor of $|\text{Out}(H_0)|$ which divides $a^2 \gcd(2, p) \gcd(2, q - 1)$. Then (4.1) implies that

$$v = \frac{cq^{36}(q^4 + 1)(q^2 + 1)(q^{12} - 1)}{be},$$

where $e = \gcd(2, q - 1)$, $c \mid ea$ and $b \mid ea^2 \gcd(2, p)$.

Note by Lemma 2.4(d) that k divides $bg(q)$, where $g(q) = q^{25}(q^2 - 1)^2(q^6 - 1)(q^8 - 1)(q^{12} - 1)$. Now Lemma 2.4(a) implies that k divides $\lambda(v - 1)$. By Tit's lemma 2.2, $v - 1$ is coprime to q , and so $\gcd(v - 1, (q^2 - 1)^2(q^6 - 1)(q^8 - 1)(q^{12} - 1))$ divides $f(q, e, b) = b^3 e^3 (q^2 + 1)^2 (q^4 + 1) (q^4 - q^2 + 1)$. Thus $mk = 2a^2 \lambda f(q, e, b)$ for some positive integer m . It follows from $k(k - 1) = \lambda(v - 1)$ that

$$k = \frac{m(v - 1)}{2a^2 f(q, e, b)} + 1 \quad (4.96)$$

Recall that $k \mid bg(q)$. Then, by (4.96), we must have

$$m(v - 1) + 2a^2 f(q, e, b) \mid 4a^4 f(q, e, b)g(q). \quad (4.97)$$

Since $m \geq 1$, $c \mid 2a$ and $b \mid 2a^2$, it follows from (4.97) that $q^{36}(q^4 + 1)(q^2 + 1)(q^{12} - 1) < 2^{10} a^{12} (q^2 + 1)^2 (q^4 + 1) (q^4 - q^2 + 1) g(q)$, which is impossible. \square

Lemma 4.12. *If X is not $E_8(q)$, then H is as in Table 2.*

Proof. Suppose $X = E_8(q)$. Then by Theorem 3.1 and Propositions 4.1 and 4.3, $X \cap H$ is one of the following:

- (i) $d \cdot D_8(q) \cdot d$ with $d = \gcd(2, q-1)$;
 - (ii) $d \cdot (A_1(q) \times E_7(q)) \cdot d$ with $d = \gcd(2, q-1)$;
 - (iii) $e \cdot (A_2^\epsilon(q) \times E_6^\epsilon(q)) \cdot e \cdot 2$ with $e = \gcd(3, q-1)$ and $\epsilon = \pm$.
- (i) Suppose $X \cap H = d \cdot D_8(q) \cdot d$. Then by (4.1), we have that

$$v = \frac{q^{64}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)}{(q^{10}-1)(q^8-1)(q^6-1)(q^4-1)}. \quad (4.98)$$

Note that $|\text{Out}(X)| = a$. So by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid ag(q), \quad (4.99)$$

where $g(q) = q^{56}(q^8-1) \prod_{i=1}^7 (q^{2i}-1)$. Moreover, Lemmas 2.4(a) and 2.2 imply that k divides $\gcd(v-1, g(q))$, and this divides $f(q)$, where $f(q) = (q^2-1)^8(q^2+1)^4(q^4+1)^2(q^{14}-1)$. Thus k must divide $a\lambda f(q)$, and so $mk = a\lambda f(q)$, for some positive integer m . Since $k(k-1) = \lambda(v-1)$, it follows that

$$k = \frac{m(v-1)}{af(q)} + 1. \quad (4.100)$$

Now we apply (4.99), and so

$$m(v-1) + af(q) \mid a^2 f(q)g(q). \quad (4.101)$$

Then $ma^2 f(q)g(q) = a^2 h(q)[m(v-1) + af(q)] + ma^2 d(q) - a^3 f(q)h(q)$, where

$$\begin{aligned} h(q) &= q^{36} - 4q^{34} + 2q^{32} + 10q^{30} - 14q^{28} + 6q^{26} + 2q^{24} - 30q^{22} + 42q^{20} + 3q^{18} - \\ &\quad 28q^{16} + 38q^{14} - 44q^{12} - 14q^{10} + 41q^8 - 33q^6 + 16q^4 + 30q^2 - 3, \\ d(q) &= 16q^{126} - 18q^{124} - 20q^{122} - 38q^{120} - 29q^{118} - 49q^{116} - 73q^{114} - 76q^{112} - 78q^{110} - \\ &\quad 118q^{108} - 131q^{106} - 146q^{104} - 151q^{102} - 173q^{100} - 173q^{98} - 185q^{96} - 158q^{94} - \\ &\quad 176q^{92} - 155q^{90} - 158q^{88} - 142q^{86} - 117q^{84} - 111q^{82} - 109q^{80} - 69q^{78} - \\ &\quad 59q^{76} - 44q^{74} - 40q^{72} - 17q^{70} - 16q^{68} - 18q^{66} - 11q^{64} + 7q^{62} + 3q^{60} - 4q^{58} + \\ &\quad q^{56} + q^{36} - 4q^{34} + 2q^{32} + 10q^{30} - 14q^{28} + 6q^{26} + 2q^{24} - 30q^{22} + 42q^{20} + \\ &\quad 3q^{18} - 28q^{16} + 38q^{14} - 44q^{12} - 14q^{10} + 41q^8 - 33q^6 + 16q^4 + 30q^2 - 3. \end{aligned}$$

Thus by (4.101), we conclude that $m(v-1) + af(q)$ divides $ma^2 d(q) - a^3 f(q)h(q)$, and so $m(v-1) < ma^2 d(q) - a^3 f(q)h(q) < ma^2 d(q)$. Thus

$$\left(\frac{1}{16}\right)q^2 < \frac{v-1}{d(q)} < a^2,$$

it follows that $q \in \{2, 3, 4, 8\}$. It is not difficult to check that for these values of $q = p^a$, $k(k-1) = \lambda(v-1)$ does not hold.

- (ii) Suppose $X \cap H = d \cdot (A_1(q) \times E_7(q)) \cdot d$. Then by (4.1), we have that

$$v = \frac{q^{56}(q^{30}-1)(q^{24}-1)(q^{20}-1)}{(q^{10}-1)(q^6-1)(q^2-1)}.$$

Note that $|\text{Out}(X)| = a$. Then Lemmas 2.4(a) and 2.1(b) imply that

$$k \mid ag(q), \quad (4.102)$$

where $g(q) = q^{64}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)^2$. Since also k divides $\lambda(v-1)$ and the fact that $\gcd(v-1, g(q))$ divides $f(q) = (q^2-1)^7(q^{12}+$

$q^6 + 1)(q^{14} - 1)$, we conclude that $k \mid a\lambda f(q)$. Thus $mk = a\lambda f(q)$, for some positive integer m . As $k(k - 1) = \lambda(v - 1)$, we have that

$$k = \frac{m(v - 1)}{af(q)} + 1. \quad (4.103)$$

Note by (4.102) that $k \mid ag(q)$. Then, by (4.103), we must have

$$m(v - 1) + af(q) \mid a^2 f(q)g(q). \quad (4.104)$$

Let now

$$\begin{aligned} h(q) &= q^{64} - 10q^{62} + 45q^{60} - 121q^{58} + 219q^{56} - 289q^{54} + 304q^{52} - 291q^{50} + 294q^{48} - \\ &\quad 332q^{46} + 410q^{44} - 544q^{42} + 737q^{40} - 934q^{38} + 1046q^{36} - 1038q^{34} + 959q^{32} - \\ &\quad 877q^{30} + 832q^{28} - 848q^{26} + 923q^{24} - 1001q^{22} + 1010q^{20} - 936q^{18} + 817q^{16} - \\ &\quad 667q^{14} + 464q^{12} - 228q^{10} + 41q^8 + 60q^6 - 144q^4 + 278q^2 - 419, \\ d(q) &= 486q^{110} - 8q^{108} + 519q^{106} + 403q^{104} + 559q^{102} + 943q^{100} + 909q^{98} + 1066q^{96} + \\ &\quad 1301q^{94} + 1591q^{92} + 1283q^{90} + 2094q^{88} + 1208q^{86} + 2173q^{84} + 1678q^{82} + \\ &\quad 1598q^{80} + 1831q^{78} + 1417q^{76} + 1533q^{74} + 1264q^{72} + 1141q^{70} + 769q^{68} + \\ &\quad 1107q^{66} + 327q^{64} + 634q^{62} + 330q^{60} + 20q^{58} + 638q^{56} - 289q^{54} + 304q^{52} - \\ &\quad 291q^{50} + 294q^{48} - 332q^{46} + 410q^{44} - 544q^{42} + 737q^{40} - 934q^{38} + 1046q^{36} - \\ &\quad 1038q^{34} + 959q^{32} - 877q^{30} + 832q^{28} - 848q^{26} + 923q^{24} - 1001q^{22} + 1010q^{20} - \\ &\quad 936q^{18} + 817q^{16} - 667q^{14} + 464q^{12} - 228q^{10} + 41q^8 + 60q^6 - 144q^4 + 278q^2 - 419. \end{aligned}$$

Then $ma^2 f(q)g(q) - a^2 h(q) \cdot [m(v - 1) + af(q)] = ma^2 d(q) - a^3 f(q)h(q)$, and so (4.104) yields $m(v - 1) + af(q) \leq ma^2 d(q) - a^3 f(q)h(q) < ma^2 d(q)$. Thus

$$\frac{q^2}{490} < \frac{(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)}{q^{54}(q^{10} - 1)(q^6 - 1)(q^2 - 1)} \leq a^2, \quad (4.105)$$

and so the inequality (4.105) holds for $q = p^a$ as in Table 19. Recall that k is a

TABLE 22. Possible values for p and a in Lemma 4.12(i).

p	2	3	5	7, ..., 19
$a \leq$	7	4	2	1

divisor of $ag(q)$. Then, for each $q = p^a$ as above, we obtain possible values of v and k , and hence for such v and k , $k(k - 1) = \lambda(v - 1)$ does not hold for any positive integer λ , which is a contradiction.

(iii) Suppose $X \cap H = e \cdot (A_2^\epsilon(q) \times E_6^\epsilon(q)) \cdot e \cdot 2$. Then by (4.1) we have that

$$v = \frac{q^{81}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)}{2 \cdot (q^9 - \epsilon)(q^6 - 1)(q^5 - \epsilon)(q^3 - \epsilon)(q^2 - 1)}.$$

Here $|\text{Out}(X)| = a$, and so by Lemmas 2.4(a) and 2.1(b), we have that

$$k \mid 2ag(q), \quad (4.106)$$

where $g(q) = q^{39}(q^{12} - 1)(q^9 - \epsilon)(q^8 - 1)(q^6 - 1)(q^5 - \epsilon)(q^3 - \epsilon)(q^2 - 1)^2$. Note by Lemma 2.4(a) that k divides $\lambda(v - 1)$, and so by Tit's lemma 2.2, k must divide $2a\lambda f(q)$, where $f(q) = (q^{12} - 1)(q^9 - \epsilon)(q^8 - 1)(q^6 - 1)(q^5 - \epsilon)(q^3 - \epsilon)(q^2 - 1)^2$. Then $mk = 2a\lambda f(q)$, for some positive integer m . As $k(k - 1) = \lambda(v - 1)$, we conclude that $2a\lambda f(q)(k - 1) = m\lambda(v - 1)$. Therefore

$$k = \frac{m(v - 1)}{2af(q)} + 1, \quad (4.107)$$

and so by (4.106), we must have

$$m(v-1) + 2af(q) \mid 4a^2 f(q)g(q). \quad (4.108)$$

Therefore $m(v-1) + 2af(q) \leq 4a^2 f(q)g(q)$, and so $q^{10} < 4a^2$, which is a contradiction. \square

ACKNOWLEDGEMENTS

The authors would like to thank anonymous referees for providing us helpful and constructive comments and suggestions. The first author would like to thank IPM (Institute for Research in Fundamental Sciences) for financial support.

REFERENCES

- [1] S. Alavi, M. Bayat, and A. Daneshkhah. Symmetric designs admitting flag-transitive and point-primitive automorphism groups associated to two dimensional projective special groups. *Designs, Codes and Cryptography*, pages 1–15, 2015. 1, 4
- [2] S. H. Alavi and M. Bayat. Flag-transitive point-primitive symmetric designs and three dimensional projective special linear groups. *Bulletin of Iranian Mathematical Society (BIMS)*, 42(1):201–221, 2016. 1, 4
- [3] S. H. Alavi and T. C. Burness. Large subgroups of simple groups. *J. Algebra*, 421:187–233, 2015. 3, 6, 7, 8, 9, 10
- [4] T. C. Burness, M. W. Liebeck, and A. Shalev. Base sizes for simple groups and a conjecture of Cameron. *Proc. Lond. Math. Soc. (3)*, 98(1):116–162, 2009. 9
- [5] P. J. Cameron and C. E. Praeger. Constructing flag-transitive, point-imprimitive designs. *J. Algebraic Combin.*, 43(4):755–769, 2016. 3
- [6] A. R. Camina. A survey of the automorphism groups of block designs. *J. Combin. Des.*, 2(2):79–100, 1994. 1
- [7] A. M. Cohen, M. W. Liebeck, J. Saxl, and G. M. Seitz. The local maximal subgroups of exceptional groups of Lie type, finite and algebraic. *Proc. London Math. Soc. (3)*, 64(1):21–48, 1992. 8, 10
- [8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. 7
- [9] B. N. Cooperstein. Maximal subgroups of $G_2(2^n)$. *Journal of Algebra*, 70(1):23 – 36, 1981. 7
- [10] A. Daneshkhah and S. Zang Zarin. Flag-transitive point-primitive symmetric designs widesigns and three dimensional projective. *Bulletin of the Korean Mathematical Society, (Accepted on December 26, 2016)*. 1
- [11] J. D. Dixon and B. Mortimer. *Permutation groups*, volume 163 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996. 4
- [12] H. Dong and S. Zhou. Alternating groups and flag-transitive 2 -($v, k, 4$) symmetric designs. *J. Combin. Des.*, 19(6):475–483, 2011. 3
- [13] H. Dong and S. Zhou. Affine groups and flag-transitive triplanes. *Sci. China Math.*, 55(12):2557–2578, 2012. 3
- [14] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.9*, 2015. 3
- [15] D. Hughes and F. Piper. *Design Theory*. Up (Methuen). Cambridge University Press, 1988. 1, 4
- [16] C. Jansen, K. Lux, R. Parker, and R. Wilson. *An atlas of Brauer characters*, volume 11 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1995. Appendix 2 by T. Breuer and S. Norton, Oxford Science Publications. 7
- [17] W. M. Kantor. Primitive permutation groups of odd degree, and an application to finite projective planes. *J. Algebra*, 106(1):15–45, 1987. 3
- [18] P. Kleidman and M. Liebeck. *The subgroup structure of the finite classical groups*, volume 129 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990. 6

- [19] P. B. Kleidman. The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups. *J. Algebra*, 117(1):30–71, 1988. [7](#)
- [20] P. B. Kleidman. The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups. *J. Algebra*, 115(1):182–199, 1988. [7](#)
- [21] P. B. Kleidman and R. A. Wilson. The maximal subgroups of $E_6(2)$ and $\text{Aut}(E_6(2))$. *Proceedings of the London Mathematical Society*, s3-60(2):266–294, 1990. [7](#)
- [22] E. S. Lander. *Symmetric designs: an algebraic approach*, volume 74 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1983. [4](#)
- [23] M. Law, C. E. Praeger, and S. Reichard. Flag-transitive symmetric 2-(96, 20, 4)-designs. *J. Combin. Theory Ser. A*, 116(5):1009–1022, 2009. [3](#)
- [24] R. Lawther. Sublattices generated by root differences. *J. Algebra*, 412:255–263, 2014. [8](#)
- [25] M. W. Liebeck and J. Saxl. On the orders of maximal subgroups of the finite exceptional groups of Lie type. *Proc. London Math. Soc. (3)*, 55(2):299–330, 1987. [9](#)
- [26] M. W. Liebeck, J. Saxl, and G. Seitz. On the overgroups of irreducible subgroups of the finite classical groups. *Proc. Lond. Math. Soc.*, 50(3):507–537, 1987. [4](#)
- [27] M. W. Liebeck, J. Saxl, and G. M. Seitz. Subgroups of maximal rank in finite exceptional groups of Lie type. *Proc. London Math. Soc. (3)*, 65(2):297–325, 1992. [7](#), [8](#), [10](#)
- [28] M. W. Liebeck, J. Saxl, and D. M. Testerman. Simple subgroups of large rank in groups of Lie type. *Proc. London Math. Soc. (3)*, 72(2):425–457, 1996. [8](#)
- [29] M. W. Liebeck and G. M. Seitz. Maximal subgroups of exceptional groups of Lie type, finite and algebraic. *Geom. Dedicata*, 35(1-3):353–387, 1990. [3](#), [7](#)
- [30] M. W. Liebeck and G. M. Seitz. On finite subgroups of exceptional algebraic groups. *J. Reine Angew. Math.*, 515:25–72, 1999. [8](#), [9](#), [10](#)
- [31] M. W. Liebeck and G. M. Seitz. The maximal subgroups of positive dimension in exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 169(802):vi+227, 2004. [7](#), [8](#), [10](#)
- [32] M. W. Liebeck and A. Shalev. The probability of generating a finite simple group. *Geom. Dedicata*, 56(1):103–113, 1995. [8](#), [9](#), [10](#)
- [33] G. Malle. The maximal subgroups of ${}^2F_4(q^2)$. *J. Algebra*, 139(1):52–69, 1991. [7](#)
- [34] S. Norton and R. Wilson. The maximal subgroups of $F_4(2)$ and its automorphism group. *Communications in Algebra*, 17(11):2809–2824, 1989. [7](#)
- [35] E. O’Reilly-Regueiro. *Flag-transitive symmetric designs*. PhD thesis, University of London, 2003. [3](#)
- [36] E. O’Reilly-Regueiro. Biplanes with flag-transitive automorphism groups of almost simple type, with alternating or sporadic socle. *European J. Combin.*, 26(5):577–584, 2005. [3](#)
- [37] E. O’Reilly-Regueiro. On primitivity and reduction for flag-transitive symmetric designs. *J. Combin. Theory Ser. A*, 109(1):135–148, 2005. [3](#)
- [38] E. O’Reilly-Regueiro. Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle. *J. Algebraic Combin.*, 26(4):529–552, 2007. [3](#), [21](#)
- [39] E. O’Reilly-Regueiro. Biplanes with flag-transitive automorphism groups of almost simple type, with exceptional socle of Lie type. *J. Algebraic Combin.*, 27(4):479–491, 2008. [2](#), [3](#), [21](#)
- [40] C. E. Praeger and S. Zhou. Imprimitivity flag-transitive symmetric designs. *J. Combin. Theory Ser. A*, 113(7):1381–1395, 2006. [3](#)
- [41] J. Saxl. On finite linear spaces with almost simple flag-transitive automorphism groups. *J. Combin. Theory Ser. A*, 100(2):322–348, 2002. [2](#), [3](#), [11](#)
- [42] G. M. Seitz. Flag-transitive subgroups of Chevalley groups. *Ann. of Math. (2)*, 97:27–56, 1973. [4](#)
- [43] M. Suzuki. On a class of doubly transitive groups. II. *Ann. of Math. (2)*, 79:514–589, 1964. [7](#)
- [44] D. Tian and S. Zhou. Flag-transitive point-primitive symmetric (v, k, λ) designs with λ at most 100. *J. Combin. Des.*, 21(4):127–141, 2013. [1](#)
- [45] D. Tian and S. Zhou. Flag-transitive 2-(v, k, λ) symmetric designs with sporadic socle. *Journal of Combinatorial Designs*, 2014. Available at <http://dx.doi.org/10.1002/jcd.21385>. [1](#)
- [46] D. Tian and S. Zhou. Classification of flag-transitive primitive symmetric (v, k, λ) designs with $\text{PSL}(2, q)$ as socle. *J. Math. Res. Appl.*, 36(2):127–139, 2016. [1](#)
- [47] R. A. Wilson. *The finite simple groups*, volume 251 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2009. [7](#)

- [48] S. Zhou and H. Dong. Sporadic groups and flag-transitive triplanes. *Sci. China Ser. A*, 52(2):394–400, 2009. [3](#)
- [49] S. Zhou and H. Dong. Alternating groups and flag-transitive triplanes. *Des. Codes Cryptogr.*, 57(2):117–126, 2010. [3](#)
- [50] S. Zhou and H. Dong. Exceptional groups of Lie type and flag-transitive triplanes. *Sci. China Math.*, 53(2):447–456, 2010. [2](#), [3](#)
- [51] S. Zhou, H. Dong, and W. Fang. Finite classical groups and flag-transitive triplanes. *Discrete Math.*, 309(16):5183–5195, 2009. [3](#)
- [52] S. Zhou and D. Tian. Flag-transitive point-primitive $2-(v, k, 4)$ symmetric designs and two dimensional classical groups. *Appl. Math. J. Chinese Univ. Ser. B*, 26(3):334–341, 2011. [3](#)

SEYED HASSAN ALAVI, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BU-ALI SINA UNIVERSITY, HAMEDAN, IRAN.

SEYED HASSAN ALAVI, SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O. BOX: 19395-5746, TEHRAN, IRAN

E-mail address: alavi.s.hassan@basu.ac.ir and alavi.s.hassan@gmail.com (G-mail is preferred)

MOHSEN BAYAT, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BU-ALI SINA UNIVERSITY, HAMEDAN, IRAN.

E-mail address: mohsen0sayeq24@gmail.com

ASHARF DANESHKHAH, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BU-ALI SINA UNIVERSITY, HAMEDAN, IRAN.

E-mail address: adanesh@basu.ac.ir